# ON PARTIALLY HYPOELLIPTIC OPERATORS. PART I: DIFFERENTIAL OPERATORS

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Abstract. This article gives a fundamental discussion on variable coefficients, self-adjoint, formally partially hypoelliptic differential operators. A generalization of the results to pseudo differential operators is given in a following article in ArXiv. Close to ([17]), we give a construction and estimates of a fundamental solution to the operator in a suitable topology. We further give estimates of the corresponding spectral kernel.

## 1. Preliminaries

For elliptic operators $P(D)$ with constant coefficients, we have the following result ([12]):

**Proposition 1.1** (Weyl's lemma). *In order for a distribution $\varphi$ defined in an open set to be an infinitely differentiable function, it is necessary and sufficient that $P(D)\varphi$ is an infinitely differentiable function in this open set.*

The hypoelliptic operators can be characterized as the class of operators, for which this proposition holds (see Def. 1.2). We will refer to this proposition, as Weyl's criterion for hypoellipticity. For these operators, the fundamental solutions are infinitely differentiable outside the origin.

The singular support of a distribution $u \in \mathcal{D}'(\ \Omega)$, sing supp u, is defined as the set of points in $\Omega$ which do not have a neighborhood in which $u$ is an infinitely differentiable function. Note that hypoellipticity in general is very sensitive to change of topology, however we will consider,

**Definition 1.2** (Hypoelliptic constant coefficients operator). *An operator $P(D)$ is said to be hypoelliptic if and only if for every open set $\Omega \subset \mathbf{R}^n$ and $\varphi \in \mathcal{D}'(\ \Omega)$ we have*

$$\textit{sing supp } P(D)\varphi = \textit{sing supp } \varphi$$

We use the term homogeneously hypoelliptic for an operator such that $P(D)u = 0$ in $\mathcal{D}'(\ \Omega)$ implies $u \in C^\infty(\Omega)$. There are several analogues to hypoellipticity that will be dealt with and several condition on an operator which are equivalent with hypoellipticity as in Weyl's criterion (see [8] sec. 11.1), and we will just mention one of these:

**Proposition 1.3** ([12],Th.II.1.2). *For an operator $P(D)$ to be hypoelliptic, it is necessary and sufficient that the polynomial $P(\xi)$ does not vanish outside a compact set and that for every multi-index $\alpha$ such that $\alpha \neq 0$ we have*

$$\lim_{\xi\to\infty} \frac{P^{(\alpha)}(\xi)}{P(\xi)} = 0.$$

A consequence of this proposition is that the polynomials corresponding to hypoelliptic operators, have the property of slow oscillation, that is

$$\frac{P(\xi+\eta)}{P(\xi)} = \sum_{\alpha,\beta} c_{\alpha,\beta} \frac{P^{(\alpha)}(\xi)}{P(\xi)} P^{(\beta)}(\eta) \to 1 \quad \xi \to \infty$$

We will also frequently use for the following proposition:

**Proposition 1.4** ([12],Prop.II.1.5)**.** *If $P(D)$ is a hypoelliptic operator, then there are two positive numbers, $c$ and $C$, such that*

$$\mid P^{(\alpha)}(\xi) \mid^2 \leq C(1+ \mid \xi \mid^2)^{-c}(1+ \mid P(\xi) \mid^2)$$

*for all $\xi \in \mathbf{R}^n$ and for every multi-index $\alpha$, $\mid \alpha \mid \neq 0$,*

Given a hypoelliptic operator $P(D)$, we will say that the operator $Q(D)$ is weaker than $P(D)$, $Q \prec P$, if $Q(\xi)/P(\xi)$ is bounded by a constant in $\mathbf{R}^n-$ infinity. We will say that $Q(D)$ is strictly weaker than $P(D)$, $Q \prec\prec P$, if the quotient $\frac{Q}{P}$ tends to 0, when $\xi \to \infty$ in $\mathbf{R}^n$. According to proposition 1.3 a condition equivalent with hypoellipticity is that $P^{(\alpha)}(\xi) \prec\prec P(\xi)$, $\alpha \neq 0$. A condition equivalent with ellipticity for a differential operator $P$ of order $m$ is that it is stronger than any differential operator of order not exceeding that of $P$ ([8]).

For variable coefficients operators, we use the same criterion for hypoellipticity as in Def. 1.2,

**Definition 1.5** (Hypoelliptic variable coefficients operator)**.** *Given an open set $\Omega \subset \mathbf{R}^n$, a differential operator $P(x,D)$ with infinitely differentiable coefficients in $\Omega$, is said to be hypoelliptic in $\Omega$, if it has the property that sing supp $P(x,D)u =$ sing supp $u$, for every $u \in \mathcal{D}'(\Omega)$.*

We will briefly indicate the techniques used in the proof of the partial analogue. The arguments will be extensively exemplified in the study.

**Definition 1.6** ( Partially regular distribution [15])**.** *A distribution $T$ is said to be regular in $x$, in an open set $\Omega \subset\subset \mathbf{R}^n \times \mathbf{R}^m$ if for every product of open sets $\Omega_x \times \Omega_y \subset \Omega$, the distribution $< T(x,y), \alpha(y) >_y$ is in $\mathcal{E}(\Omega_x)$, for every $\alpha$ in $\mathcal{D}(\Omega_y)$.*

The norm $\parallel \varphi \parallel_{s,t}$ can be defined, for $\varphi \in \mathcal{E}'$ and $s,t$ real numbers, as

$$\parallel \varphi \parallel_{s,t}^2 = \int \mid \widehat{\varphi}(\xi,\eta) \mid^2 (1+ \mid \xi \mid^2)^s (1+ \mid \eta \mid^2)^t d\xi d\eta$$

A distribution $f$ is said to be of order $(s,t)$, if for every $\alpha \in \mathcal{D}(\mathbf{R}^{n+m})$, $\parallel \alpha f \parallel_{s,t} < \infty$.

**Proposition 1.7** ([6],section 2)**.** *A distribution $f \in \mathcal{D}'(\Omega)$, $\Omega$ an open set in $\mathbf{R}^\nu$, is regular in $x$ if and only if, for every open set $W \subset\subset \Omega$ and for every real $s$, there is a number $t$, dependent on $s$ and $W$, such that $f$ is of order (s,t)*

**Definition 1.8** (Partially hypoelliptic operator [15])**.** *A differential operator $L(x,y,D_x,D_y)$, is said to be hypoelliptic in $x$ if every distribution solution $u$ to $Lu = f$ is regular in $x$, when $f$ is regular in $x$.*

Using the proposition 1.7,we see that given a distribution $f$ of order $(s,-N)$ ($f$ is assumed partially regular) and $Lu = f$ with $u$ of order $(s',-N)$, $s'$ arbitrary, to

conclude that the operator $L$ is partially hypoelliptic, we only have to prove that $u$ is of order $(s'', -N')$ for some suitable $N'$ and $s'' > s$.

We will limit our study to finite order distributions and for this reason we do not initially separate between the concept of hypoelliptic operators and homogeneously hypoelliptic operators.

Finally, for the polynomial corresponding to an operator with constant coefficients, $P(D_x, D_y)$, we can give the following equivalent conditions for partial hypoellipticity;

**Proposition 1.9** ([8], section 11.2)**.** *The following conditions are necessary and sufficient for a polynomial to be hypoelliptic in $x$*

i) $P^{(\alpha)}(\xi, \eta)/P(\xi, \eta) \to 0$ *if* $\alpha \neq 0$ *and* $\xi \to \infty$ *while* $\eta$ *remains bounded.*

ii) *$P$ can be written as a finite sum*

$$P(\xi, \eta) = \sum_{\alpha'=0} P_\alpha(\xi)\eta^\alpha$$

*where $P_0(\xi)$ is hypoelliptic (as a polynomial in $\xi$) and $P_\alpha(\xi)/P_0(\xi) \to 0$ when $\xi \to \infty$ if $\alpha \neq 0$.*

## 2. Mizohata's representation

For polynomials over $\mathbf{R}^n$, $P, Q$ with constant coefficients, assuming both are HE (hypoelliptic), we can define the following equivalence-relation: $P \sim Q$ if there exist $C, C'$ positive constants such that for all $\xi \in \mathbf{R}^n$

$$C \leq \frac{1+ \mid P(\xi) \mid^2}{1+ \mid Q(\xi) \mid^2} \leq C'$$

Assuming that $P$ is partially hypoelliptic, we will later show, that it is sufficient to consider the real and imaginary parts of the operator separately. We will thus, if nothing else is indicated, assume that the operators we are considering are self-adjoint. Note that for constant coefficients operators we always have $P \sim P^*$. An operator $L(x, y, D_x, D_y)$ with coefficients in $C^\infty(\Omega), \Omega$ an open set in $\mathbf{R}^{n+m}$ is called partially formally hypoelliptic (PFHE) of type $M$, where $M = M(D_x)$ is hypoelliptic, if for $(a, b) \in W$, $W$ a neighborhood of $(x^0, y^0) \in \Omega$, the contact operator (the frozen operator)

$(*)$ $\qquad L(a, b, D_x, D_y) \sim_x M(D_x)$

The notation $\sim_x$ will be explained below. We note that since $M$ is assumed HE, this criterion implies that $L(a, b, D_x, D_y)$ is HE in $x$. The class of operators equivalent with a given operator constitute a finite dimensional vector space, (after adding 0 to the class) . This means that $M$ has the representation $M(D_x) = \sum_{j=1}^r M_j(D_x)$, where $M_j(D_x) \sim M(D_x)$ for every $j$ and $r$ a finite integer. Further, for a constant coefficients operator $P(D_x) = \sum_{k=1}^r N_k(D_x)$ such that $N_k(D_x) \sim M(D_x)$ for every $k$, we must have $M_j \sim N_k$, for every $j, k$.

**Lemma 2.0.1.** *Assume that $M(D_x)$ is a constant coefficients hypoelliptic operator and that $\mathbf{P} = (P_1, \ldots, P_r)$ is a vector of constant coefficients operators equivalent in strength with $M$. Assume $P = \sum_{j=1}^r c_j P_j$, $c_j$ in $\mathbf{C}$ for all $j$, is such that for all $j$, $P_j$ is weaker than $P$. Then $P$ and $M$ are equally strong.*

Proof:

Let $P^{-2} = (\sum_j \mid P_j \mid^2)$. Clearly $P \prec P^-$ and $P^- \prec M$. Since

$$\mid \frac{M(\xi)}{P_j(\xi)} \frac{P_j(\xi)}{P^-(\xi)} \mid < C$$

for large $\mid \xi \mid$, we have $P^- \sim M$. Finally

$$\mid \frac{M(\xi)}{P_j(\xi)} \frac{P_j(\xi)}{P(\xi)} \mid < C$$

for $\mid \xi \mid$ large. $\square$

**Remark** : Assume $P = \sum_{j=0}^r P_j$, a decomposition in hypoelliptic operators, such that $P_j$ is weaker than $P$ for every $j$. If there is an operator in this development, say $P_0$, such that $P_0 \sim P^-$, then $P_0 \prec P \prec P^-$, that is $P \sim P^-$ and $P$ is hypoelliptic.

The criterion $(*)$ should now be understood using the representation

$$L(a, b, D_x, D_y) = \sum_j c_j(a, b) N_j(D_x) Q_j(D_y)$$

where $N_j, Q_j$ are constant coefficient operators, such that $N_j$ is weaker than $L' = \sum_j c_j(a, b) N_j$, for any $j$, as

$(*')$ $\qquad N_j(D_x) \sim M(D_x) \quad$ for all $j$

L is said to be PFHE in $\Omega$ (particularly FHE in x), if the condition $(*)$ is satisfied for every $(x_0, y_0) \in \Omega$. Note that if we do not assume the polynomials $P, Q$ hypoelliptic, we can use the following criterion for equivalence found in ([8], sec. 10.3.4)

$$C < \widetilde{P}(\xi)/\widetilde{Q}(\xi) < C' \qquad \xi \in \mathbf{R}^n$$

where $\widetilde{P}(\xi)^2 = \sum_{|\alpha| \geq 0} \mid P^{(\alpha)}(\xi) \mid^2$.

**Lemma 2.0.2.** *Assume $L(\xi) = P_0(\xi') + \sum_j P_j(\xi') Q_j(\xi'')$ is the polynomial for a constant coefficients partially hypoelliptic operator, (as in Prop. 1.9 ii)) and $M(\xi') = \sum_j M_j(\xi')$ with a development in equivalent operators as in Lemma 2.0.1, the polynomial for a constant coefficients operator equivalent with $P_0$ in strength. Then $L \sim \sum_j M_j Q_j$.*

Proof:
It will turn out that the real zero's of the polynomial do not contain necessary information concerning regularity behavior to the operator and we may assume (possibly by adding a parameter and/or by squaring the expressions), that the polynomial has no real zero's, for $\xi$ large.

i) Let $P^+ = \sum_j (P_0 + P_j) Q_j$. Then $P^+ \sim \sum_j M_j Q_j$, since we trivially have

$$\frac{\mid (P_0 + P_j)(\xi') Q_j(\xi'') \mid}{\mid M_j(\xi') Q_j(\xi'') \mid} \frac{\mid M_j(\xi') Q_j(\xi'') \mid}{\mid \sum_j M_j(\xi') Q_j(\xi'') \mid} < C, \qquad \mid \xi \mid \text{ large },$$

and analogously $\sum_j M_j Q_j \prec P^+$.

ii) We have $L \sim P^+$. It immediately follows that $L \prec P^+$. The condition $M \sim P_0$, means that for $\xi''$ fixed and bounded

$$\mid \frac{M(\xi')}{L(\xi', \xi'')} \mid < C \qquad \text{for } \mid \xi' \mid \qquad \text{large}$$

so $M_j Q_j \prec L$. Finally

$$\frac{|\,(P_0+P_j)(\xi')Q_j(\xi'')\,|}{|\,M_j(\xi')Q_j(\xi'')\,|}\frac{|\,M_j(\xi')Q_j(\xi'')\,|}{|\,L(\xi)\,|} < C \qquad |\,\xi\,| \text{ large}$$

We conclude $P^+ \sim L\square$

Note also that $L(x,y,D_x,D_y)$ can be written as an operator with constant strength

$$L(x_0,y_0,D_x,D_y) + \sum_{j=1}^{r} d_j(x,y)R_j(D_x,D_y)$$

where the coefficients $d_j(x,y)$ are uniquely determined, vanish at $(x_0,y_0)$, and have the same regularity properties as the coefficients to $L(x,y,D_x,D_y)$. Further, the $R_j$'s are constant coefficients operators weaker than $M(D_x)N(D_y)$, where $M$ is the type operator and $N$ a constant coefficients operator, adjusting the lower order term behavior.

The representation of principal interest in this study is however the following, due to Mizohata (section 2,[15]):

$$(1) \qquad L(x,y,D_x,D_y) = P(x,y,D_x) + \sum_{j=1}^{r} P_j(x,y,D_x)Q_j(D_y)$$

where

(1) $P$ is FHE in $x$, in the sense that $P(x,y,D_x) = \sum_j a_j(x,y)M_j(D_x)$ in an open set $W \subset \Omega$, where $a_j \in C^\infty$ , $M_j \sim M$ and HE for all $j$ $(j = 1,\ldots,r)$.
(2) $P_j \prec\prec {}_x M$, in the sense that $P_j(x,y,D_x) = \sum_k b_k(x,y)N_k(D_x)$ in $W$, where $b_k \in C^\infty$, $N_k \prec\prec M$ for all $k$ $(k = 1,\ldots,r)$.
(3) $Q_j$ is a constant coefficients operator for all $j$.

We assume in what follows that $\deg_{(x)}(L) > 0$. We shall in this study assume that $L$ is defined according to Mizohata's representation on a fixed compact set $K \subset \Omega$ and that it is defined as the type-operator $M$, outside $K$.

**Proposition 2.0.3.** *If an operator can be represented according to Mizohata (1), then it is FHE in $x$ of type $M$. Conversely, any operator FHE in $x$ of type $M$ can be written on the form above.*

Proof: The fact that the operator with Mizohata's representation is HE in $x$, is proven in [15]. It can also be shown, that given operators $P$ and $Q$ with constant coefficients, such that $Q \prec\prec P$, we get $P \sim P + aQ$, for all $a \in \mathbf{C}$. This means that $L \sim_x M$ in $K$ for the operator $L(a,b,D_x,D_y)$ with constant coefficients. We conclude that $L$ is FHE in $x$.

It is always possible to write the polynomial corresponding to the operator $L$ as

$$(2) \qquad L(x,y,\xi,\eta) = P(x,y,\xi) + \sum_{|\alpha|>0} b_\alpha(x,y)P^{(\alpha)}(\xi)\eta^\alpha$$

For a fixed $\eta$, we know that $L$ must be FHE in $x$ and of type $M$, particularly if $\eta = 0$, $P(x,y,\xi)$ must be FHE in $x$. According to the definition, this implies that $P(a,b,\xi)$ is HE in $x$, for a fixed $(a,b)$ in $K$, consequently $P^{(\alpha)} \prec\prec M$ and finally $b_\alpha(a,b)P^{(\alpha)}(\xi)\eta^\alpha \prec\prec {}_x M(\xi).\square$

## 3. Generalized Sobolev spaces

We start with a generalization of the Sobolev spaces, written as $H(Q_1,\ldots,Q_r;\Omega)$, where $u \in L_2(\Omega)$ and in the distributional sense, $Q_i u \in L_2(\Omega)$,for all $i$. Let $\mathbf{Q}$ stand for the constant coefficient operators $(Q_1,\ldots,Q_r)$. This produces separable Hilbert spaces $H(\mathbf{Q},\Omega)$, for the norm

$$\| u \|^2_{H(\mathbf{Q},\Omega)} = \| u \|^2_{L_2(\Omega)} + \sum_{i=1}^{r} \| Q_i u \|^2_{L_2(\Omega)} \qquad u \in L_2(\Omega).$$

Using Parseval's relation and applying weight functions to these spaces, we get Hilbert spaces of compactly supported distributions, $H^{s,t}_K(\mathbf{Q})$, $H^t_K(\mathbf{Q})$, for real numbers $s,t$. The same construction holds for variable coefficients operators and we get Hilbert spaces $H^s_K(\mathbf{P})$, short for $H^s_K(P_1,\ldots,P_r)$, with the norm

$$\| u \|^2_{H^s_K(\mathbf{P})} = \| u \|^2_s + \sum_{j=1}^{r} \| P_j(x,D_x)u \|^2_s \qquad u \in H^s_K \tag{3}$$

Proof:([11])
It is sufficient to study the case $s = t = 0$. The right side in (3) is a pre-Hilbert scalar product. If the left side is 0, then $\| u \|^2_{L^2} = 0$, so $u = 0$. Obviously, $L^2_K(\mathbf{P}) \subset L^2_K$ algebraically. Also if $\varphi \to 0$ in $L^2_K(\mathbf{P})$, then $\varphi \to 0$ in $L^2_K$, so the inclusion holds topologically. There remains to prove completeness for the norm (3). Assume $\varphi_k$ a Cauchy sequence in $L^2_K(\mathbf{P})$, then $\varphi_k, P_i\varphi_k$ are Cauchy sequences in $L^2_K$, which is a complete space, so $\varphi_k \to \varphi$ and $P_i\varphi_k \to \psi_i$ in $L^2_K$. Further, $P_i\varphi_k \to P_i\varphi$ in $\mathcal{D}'$ and we conclude that $\psi_i = P_i\varphi$. $\square$

Note that it follows from the proof of Lemma 2.0.1, that if we assume the operators $P_j$ hypoelliptic for all $j$, then the condition that $P = \sum_j P_j$ ( a development in weaker operators ) is hypoelliptic, is equivalent with the condition that $P \sim P^- = (\sum_j \mid P_j \mid^2)^{1/2}$. Particularly, if $P$ is hypoelliptic then the corresponding norms are equivalent.

We now consider the operator as acting on the Hilbert spaces $H^{s,t}_K$, where $s,t$ are real numbers, $K$ is a compact set and

$$H^{s,t}_K = \{ f \in \mathcal{E}'(K) \quad (1+\mid \xi \mid^2)^{s/2}(1+\mid \eta \mid^2)^{t/2}\hat{f} \in L^2_{\xi,\eta} \}$$

$H^{s,t}_\Omega$ is defined as the inductive limit as $K$ varies in $\Omega$. We also consider the Fréchet spaces $H^{s,t}_{loc}(\Omega) = \{ f \in \mathcal{S}'(\Omega);\quad \varphi f \in H^{s,t}_\Omega \text{ , for all } \varphi \in C^\infty_0(\Omega) \}$. According to ([15]), any distribution $T$ in $H^{s,t}$ can be mollified with a tensor product of test functions, that is for $\alpha,\beta \in \mathcal{D}$ non-negative with $\int \alpha(x)dx = \int \beta(y)dy = 1$, $\alpha_\epsilon(x) = \epsilon^{-n}\alpha(x/\epsilon)$ and $\beta_\epsilon(y) = \epsilon^{-m}\beta(y/\epsilon)$, we have $\alpha_\epsilon \otimes \beta_\epsilon * T \to T$ in $H^{s,t}$, as $\epsilon \to 0$. We can also use a partial mollifier, that is $T *' \alpha_\epsilon(x,y) \to T$ in $H^{s,t}$. An argument similar to the proof above now gives a generalized Sobolev norm, corresponding to the variable coefficients operator P in (1), formally hypoelliptic in $x$.

$$\| u \|^2_{H^{s,t}_K(P)} = \| u \|^2_{s,t} + \sum_{j=1}^{r} \| P_j(x,y,D_x)u \|^2_{s,t} \qquad u \in H^{s,t}_K$$

The norm corresponding to the Mizohata's representation (1) becomes

$$\| u \|^2_{H^{s,-N}_K(L)} = \| u \|^2_{s,-N} + \| P(x,y,D_x)u \|^2_{s,-N} + \sum_j \| P_j(x,y,D_x)Q_j(D_y)u \|^2_{s,-N}$$

for $u \in H^{s,-N}_K$

3.1. **The spaces $H_K^{s,-N}$.** We will use the notation ([12]) $H_K^{p_s,t}$, for $\{f \in \mathcal{E}'(K); p_s(\xi)\ (1+ \mid \eta \mid^2)^{t/2} \hat{f} \in L^2_{\xi,\eta}\}$, where $p_s$ is a weight function over $\mathbf{R}^n$.

**Lemma 3.1.1.** *We have the following equivalence for the weight function $p_s(\xi) = (1+ \mid \xi \mid^2)^{s/2}(1+ \mid M(\xi) \mid^2)$ : For the operator $P(x, y, D_x)$ FHE in $x$ of type $M(D_x)$, there is a compact neighborhood $K$ of $(x_0, y_0)$, such that*

$$f \in H_K^{s,-N}(P) \ \Leftrightarrow\ f \in H_K^{p_s,-N}$$

*$s$ is here assumed real and $N$ integer.*

Proof:([12] Lemma III.1.3)
Let $W$ be a compact neighborhood of $(x^0, y^0)$ included in $K$. The conditions on $P$ allow us to write $P(x, y, D_x) = \sum_i a_i(x, y)P_i(D_x)$ with $a_i \in C^\infty$ and $P_i \sim M$, $P_i$ with constant coefficients. For the implication $f \in H_K^{p_s,-N} \Rightarrow P_i(D_x)f \in H_W^{s,-N}$, we use the trivial inequality
(**) $\qquad \parallel Mf \parallel_{s,-N} \leq C_K \parallel f \parallel_{p_s,-N}.$
From the condition $P_i \sim M$ and consequently $P_i \prec M$, we conclude that ([15] Cor. Lemma 2.2), $\parallel P_i(D_x)f \parallel_{s,-N} \leq C_K \parallel M(D_x)f \parallel_{s,-N}$. Thus $P_i(D_x)f \in H_W^{s,-N}$. Finally, $H_W^{s,-N}$ is a space of local type ([15] Cor.2 Prop. 2.1), which means that $P_i(D_x)f \in H_W^{s,-N} \Rightarrow a_i(x, y)P_i(D_x)f \in H_W^{s,-N}$, for all $a_i \in C^\infty$. This establishes the implication $f \in H_K^{p_s,-N} \Rightarrow P(x, y, D_x)f \in H_K^{s,-N}$. The opposite implication is proven in [15] Prop. 2.4 for the local spaces, but with the additional assumption that $f$ has compact support, the result follows on $K$ using an appropriate test function.□

We also note, that the same implications can be established for the operator $L$, on the local spaces ([12] Theorem III.1.4), although this requires an adjustment of the order in the "bad" variable. Thus

$$f \in H_K^{s,-N} \quad \text{and} \quad L(x, y, D_x, D_y)f \in H_K^{s,-N'} \Leftrightarrow\ f \in H_K^{p_s,-N}$$

3.2. **Sobolev's embedding theorem.** In section 3.1, we proved that $\parallel \cdot \parallel_{H_K^{s,-N}(P)}$ is equivalent to $\parallel \cdot \parallel_{H_K^{p_s,-N}}$, where in the last norm $P$ denotes the single operator with variable coefficients formally hypoelliptic in $x$, according to section 2. If $Mf \in H_K^{s,-N}$, then by the regularity property of Sobolev spaces, we have $f \in H_K^{s+\sigma,-N}$, for some positive number $\sigma$. Thus $f \in H_K^{s,-N}(M)$. Immediately $H_K^{s,-N}(M) = H_K^{p_s,-N}$. We conclude that we have the following norm equivalences, $\parallel \cdot \parallel_{H_K^{s,-N}(M)} \sim \parallel \cdot \parallel_{H_K^{p_s,-N}} \sim \parallel M \cdot \parallel_{H_K^{s,-N}}$. We note in particular:

$$\parallel Mf \parallel_{s,-N} \leq C_K \parallel f \parallel_{H_K^{s,-N}(P)} \qquad f \in H_K^{s,-N} \tag{4}$$

It can easily be proved that for an operator $L$, partially formally hypoelliptic in $x$ and of type $M$, the iterated operator $L^r$, $r$ any integer, is partially formally hypoelliptic in $x$ and of type $M^r$. It thus admits representation

$$L^r(x, y, D_x, D_y) = P_{(r)}(x, y, D_x) + \sum_j P_{(r),j}(x, y, D_x)Q_{(r),j}(D_y)$$

and a direct calculation shows that $P_{(r)} = P^r$, the iterated operator FHE in $x$ being of type $M^r$.

The inequality (4) above now becomes

$$\parallel M^r f \parallel_{s,-N} \leq C_{K,r} \parallel f \parallel_{H_K^{s,-N}(P^r)} \qquad f \in H_K^{s,-N}$$

Since $(1+ \mid M(\xi) \mid^2)^r \leq C(1+ \mid M(\xi) \mid^{2r})$ and using the well known inequality for hypoelliptic operators (Proposition 1.4 for r=1)

$$(1+ \mid \xi \mid^2)^{kr} \leq C(1+ \mid M(\xi) \mid^2)^r \tag{5}$$

for some positive $k, C_K$ and for all $\xi \in \mathbf{R}^n$, we get

$$\parallel f \parallel_{kr+s,-N} \leq C_K \parallel f \parallel_{H_K^{s,-N}(P^r)} \qquad f \in H_K^{s,-N}$$

We can assume $k \leq 1$.

**Remark** : We could also show that the condition on $M$: $\mid M(\xi) \mid \geq C \mid \xi \mid^{\varrho}$ gives $(1+ \mid \xi \mid^2)^{\varrho} \leq C(1+ \mid M(\xi) \mid^2)$, so if $r$ is integer $(1+ \mid \xi \mid^2)^{r\varrho} \leq C(1+ \mid M(\xi) \mid^{2r})$. In what follows we assume this condition satisfied.

In the following modified Sobolev's embedding theorem, we use the notation $\mathcal{U}_{-N}$, for a tempered fundamental solution of the operator $(1-\Delta)^{N/2}$. According to Schwartz,

$$\mathcal{U}_l = \frac{\pi^{l/2}}{\Gamma(l/2)} \mathrm{Pf.} \Big[ (\frac{r}{2\pi})^{\frac{l-m}{2}} \mathrm{K}_{\frac{m-l}{2}}(r) \Big] \qquad (1-\Delta)^k \mathcal{U}_l = \mathcal{U}_{l-2k}$$

except for $l$ even integer $\leq 0$, when $\mathcal{U}_{2k} = (1-\Delta)^k \delta_0$. $K$ is an analytic function outside the origin, it is non negative for $l \geq 0$ and it is exponentially decreasing towards 0 in infinity. This means that it is in $\mathcal{O}_C$ and convolution with any distribution in $\mathcal{S}'$ is well defined. (for notation, definitions and results, see [25] chapter 30)

**Proposition 3.2.1.** *For $P$ an operator formally hypoelliptic in $x$ with coefficients in $C^\infty(\Omega)$, for $f \in H_K^{kr+s,t}$, $K$ a compact set in $\Omega$, $s,t$ real numbers and for $kr+s \geq \frac{n}{2}+l$, there is an integer $N$ and a constant $C_K$, such that*

$$\sup_{K, |\alpha| \leq l} \mid (D_x^\alpha f *'' \mathcal{U}_{-2N})(x,y) \mid \leq C_K \Big( \parallel P(x,y,D_x)^r f \parallel_{s,-N} + \parallel f \parallel_{s,-N} \Big)$$

*Here $\mathcal{U}_{-N}$ is defined as $(1-\Delta)^{-N/2}\delta_0$, the Laplacian is taken in $\mathbf{R}^m$*

Proof: In general we have $v \in L^1_\xi \Rightarrow \mathcal{F}_\eta^{-1} v \in C^0_x$ (here $\mathcal{F}_\eta$ is the partial Fourier transform acting on $x \to \xi$), so if we can show that $\mathcal{F}_\eta D_x^\alpha u(x,\eta)(\xi) \in L^1_{\xi,\eta}$, for $u = (1+ \mid \eta \mid^2)^{-N} \mathcal{F}_\xi f(\xi,\eta)$, assuming that $f \in H_K^{kr+s,-N}$, we must have that $D_x^\alpha u \in C^0_x$. Thus $D_x^\alpha f *'' \mathcal{U}_{-2N} \in C^0_{x,y}$, for $\mid \alpha \mid \leq l$ and $\mathcal{F}\mathcal{U}_{-2N} = (1+ \mid \eta \mid^2)^{-N}$. Now $\mathcal{F}_\eta D_x^\alpha u(x,\eta)(\xi) = \xi^\alpha \mathcal{F}_\eta u(\xi,\eta)$. This gives using Hölders inequality:

$$\iint (1+ \mid \eta \mid^2)^{-N/2} \mid \mathcal{F}_\eta D_x^\alpha u(x,\eta)(\xi) \mid d\xi d\eta \leq$$
$$\Big( \iint (\mid \xi \mid^\alpha)^2 (1+ \mid \xi \mid^2)^{-(kr+s)} (1+ \mid \eta \mid^2)^{-N} d\xi d\eta \Big)^{1/2}$$
$$\times \Big( \iint (1+ \mid \xi \mid^2)^{kr+s} \mid \mathcal{F}_\eta u \mid^2 d\xi d\eta \Big)^{1/2}$$

For the first integral, $\mid \xi^\alpha \mid^2 \leq (1+ \mid \xi \mid^2)^{|\alpha|} \leq (1+ \mid \xi \mid^2)^l$ and $(1+ \mid \xi \mid^2)^{kr+s-l}(1+ \mid \eta \mid^2)^{-N} \in L^1_{\xi,\eta}$, if $2(kr+s-l) > n$ and $N > m$, that is $u \in C^l_x$. For the second integral, $\mathcal{F}_\eta u = (1+ \mid \eta \mid^2)^{-N/2} \mathcal{F} f$, so the last integral is finite through the conditions on f.□

We note that the same claim holds for the local spaces. If $x_0$ is fixed in $\Omega$ and $\varphi \in C_0^\infty(\Omega)$ with $\varphi = 1$ in a neighborhood $K$ of $x_0$ and $u \in H_{loc}^{kr+s,-N}$, we must

have $u = \varphi u \in H_K^{kr+s,-N}$. The inclusion follows from the proof above and the fact that the neighborhood is arbitrary.

## 4. The operator Re $L$

We will later make use of the notion of a $\mathcal{U}_{t,-N}$-formally self-adjoint operator. By this we mean that the operator is self-adjoint in the weighted scalar product

$$(Lu, v)_{t,\text{-}N} = (u, Lv)_{t,\text{-}N} \qquad \mathcal{U}_{t,\text{-}N}(D_x, D_y) = (1-\Delta_x)^{t/2}(1-\Delta_y)^{-N/2}\delta_0$$

We introduce the notion of an operator partially formally self-adjoint, meaning that the formally hypoelliptic part of the operator is $\mathcal{U}_{t,\text{-}N}$-formally self-adjoint, that is if $u, v \in H_K^{t,-N}(Q)$, $t$ a real number and $N$ integer

$$\sum_{j=1}^{r}(P_j(x,y,D_x)Q_j(D_y)u, Q_j(D_y)v)_{t,-N} = \sum_{j=1}^{r}(Q_j(D_y)u, P_j(x,y,D_x)Q_j(D_y)v)_{t,-N}$$

assuming both sides are finite. In this study we will usually assume the operator L formally self-adjoint as well as partially formally self-adjoint. The proof of the following Lemma is close to [17] Lemma 3.

**Lemma 4.0.1.** *If L is formally hypoelliptic in x of type M as in (1) and partially formally self-adjoint, then Re L is formally hypoelliptic in x and of type M.*

Assume $L = \sum_{j=0}^{r} P_j(x,y,D_x)Q_j(D_y) = \sum_k b_k(x,y)N_k(D_x)Q_k(D_y)$,$Q_0 = I$, our partially hypoelliptic, variable coefficients operator, such that $Q_j, N_k$ are real, constant coefficients operators. Further, that $L$ is self-adjoint in $H^{s,t}(\mathbf{R}^\nu)$, that is

$$(v_{s,t}(D)L(x,y,D',D'')u, v)_{L^2} = (v_{s,t}(D)u, L(x,y,D',D'')v)_{L^2} \qquad u, v \in H^{s,t}$$

It is for our study sufficient to consider weights $v_{0,-t}(D'') = (1-\Delta_y)^t$. Assuming $PQ$ is any of the $P_jQ_j$, $j \geq 1$, in the development of $L$, we have

$$\begin{gathered} 0 = (v_{0,t}(D'')P(x,y,D')Q(D''))^* - \overline{(v_{0,t}(D'')P(x,y,D')Q(D''))} \\ + \overline{(v_{0,t}(D'')P(x,y,D')Q(D'')} - (v_{0,t}(D'')P(x,y,D')Q(D'')) = \\ T - 2i \text{ Im } (v_{0,t}(D'')P(x,y,D')Q(D'')) \end{gathered}$$

Assume further $P$ self-adjoint in $H^{s,t}$, that is $L$ partially self-adjoint. Then,

$$\begin{gathered} 0 = Q(D'')\big[P(x,y,D'), v_{0,t}(D'')\big] + \big[Q(D''), v_{0,t}(D'')\big]P(x,y,D') + \\ v_{0,t}(D'')\big[Q(D''), P(x,y,D')\big] \end{gathered}$$

where $\big[A, B\big] = AB - BA$. Obviously,
$0 = Q(D'')\big[P(x,y,D'), v_{0,t}(D'')\big] + v_{0,t}(D'')\big[Q(D''), P(x,y,D')\big]$ and

$$Q(D'')v_{0,-t}(D'')P(x,y,D')v_{0,t}(D'') = P(x,y,D')Q(D'') \tag{6}$$

Let

$$\begin{gathered} T(x,y,D) = Q(D'')\big[P(x,y,D'), v_{0,t}(D'')\big] + v_{0,t}(D'')\big[Q(D''), P(x,y,D')\big] + \\ 2iv_{0,t}(D'')(\text{ Im } P(x,y,D'))Q(D'') \end{gathered}$$

and $R(D'') = (1-\Delta'')^t Q(D'')$, then
$R(D'')P(x,y,D')v_{0,t}(D'') - P(x,y,D')Q(D'') = 0$ can be written

$$\sum_{\alpha'' \neq 0} D_y^{\alpha''} P(x,y,D')R^{(\alpha'')}(D'')v_{0,t}(D'') = 0 \tag{7}$$

The sum on the left side in (7), can be divided into two partial sums,
$\sum' + \sum'' = \sum_{m<|\alpha''|} + \sum_{m \geq |\alpha''|}$, where $m =$ deg $Q(\xi'')$. We now have

$\sum'(x,y,D) \prec\prec P(x,y,D')Q(D'')$ and since $\sum' + \sum'' = 0$, also $\sum''(x,y,D) \prec\prec P(x,y,D')Q(D'')$. Put

$$v_{0,-t}(D'')T(x,y,D) = \sum{}' (x,y,D) + \sum{}'' (x,y,D) + 2i(\text{ Im } P(x,y,D'))Q(D'').$$

Obviously, $v_{0,t}\sum', v_{0,t}\sum'' \prec\prec PQ$ and if we choose $m < 2t$, $v_{0,t}(D'')(\text{ Im } P(x,y,D'))Q(D'') \prec\prec P(x,y,D')Q(D'')$, that is $T(x,y,D) \prec\prec P(x,y,D')Q(D'')$ and we conclude

**Proposition 4.0.2.** *Assume $L(x,y,D)$ a variable coefficients, partially formally hypoelliptic operator, self-adjoint and partially self-adjoint over $H^{s,t}(\mathbf{R}^\nu)$, then Re $L(x,y,D) \sim L(x,y,D)$*

Note that with the additional assumption that $P(x,y,D')Q(D'') = Q(D'')P(x,y,D')$, we have $\text{ Im } (P(x,y,D')Q(D'')) = 0$.

We are also interested in the behavior of Re $L$ at the infinity. Consider first the type polynomial $M$. Using the inequality for hypoelliptic operators (5), we see that $\mid M(\xi) \mid \to \infty, \mid \xi \mid \to \infty$. If we assume $M$ with real coefficients and also $M(\xi) \geq 1$, we have $M(\xi) \to \infty, \mid \xi \mid \to \infty$ . We will adopt these assumptions throughout this study.

**Lemma 4.0.3.** *With $L$ as in the previous lemma and defined on an open connected set $\Omega$, we have that Re $L(x,y,\xi,\eta)$ does not change sign at the $\mathbf{R}^n-$infinity.*

Proof: First assume that $L$ has constant coefficients as an operator hypoelliptic in $x$. According to [8] (sec.11.1), if $d(\xi,\eta)$ denotes the distance from $(\xi,\eta) \in \mathbf{R}^{n+m}$ to the surface $\{(\zeta,\upsilon) \in \mathbf{C}^{n+m}; L(\zeta,\upsilon) = 0\}$, it follows that $d(\xi,\eta) \to \infty$ as $\xi \to \infty$, while $\eta$ remains bounded. Thus Re $L$ cannot change sign at the $\mathbf{R}^n$-infinity.

For the variable-coefficients case, we make use of the assumption that $\Omega$ is connected. We know that for every fixed $(x,y) \in \Omega$, $\mid$ Re $L(x,y,\xi,\eta) \mid \to \infty$, as $\xi \to \infty$, while $\eta$ bounded. Let $E\pm = \{(x,y) \in \Omega; \text{Re } L(x,y,\xi,\eta) \to \pm\infty$, $\xi \to \infty$, $\eta$ bounded$\}$. We have already proved that $E_+ \bigcap E_- = \emptyset$. Since Re $L$ is continuously dependent on $(x,y)$, the sets $E\pm$ must be open, but this would give a separation of $\Omega$, which contradicts the assumption that $\Omega$ is connected! We can without loss of generality assume that Re $L(x,y,\xi,\eta) \to \infty$, at the $\mathbf{R}^n$-infinity. $\Box$

**Lemma 4.0.4.** *If $P$ is a reduced (cf. section 11) and real operator on $L^2$, then $P^{1/N}$ is defined as an operator on $L^2$ for some positive integer $N$.*

Proof: We can assume $P(\xi) \to \infty$ as $\mid \xi \mid \to \infty$. It is sufficient to study the polynomial for $\mid \xi \mid$ large, that is if $\tilde{P}$ is the polynomial adjusted to a constant on $\mid \xi \mid \leq R$, we have $D(P_0) = D(\tilde{P}_0)$. The minimal operator, $\tilde{P}_0$, can thus be considered as a positive, selfadjoint operator on $L^2$ and the "Gelfand-Naimark-theorem" gives unique existence of a selfadjoint operator $\tilde{P}_0^{1/N}$, for some $N$.$\Box$

Assume $P = P_1 + iP_2$ homogeneously (geometrically) hypoelliptic on $\mathcal{D}'$. This means particularly that $\frac{P_1(\xi+\vartheta)}{P_1(\xi)} \to 1$ and $\frac{P_2(\xi+\vartheta)}{P_2(\xi)} \to 1$ as $\mid \xi \mid \to \infty$ and $\mid \vartheta \mid \leq A$ for some constant $A$ and $\xi, \vartheta$ real. If $P_1/P \to \alpha = 1/(1+iC)$, with $C$ a real constant, then $P_2/P_1 \to -C$ as $\mid \xi \mid \to \infty$. If $\mid \xi \mid^\sigma \leq P_1(\xi)$, $P_2(\xi) \leq \mid \xi \mid^\delta$, then $\mid P_2^{1/N}/P_1 \mid \leq \mid \xi \mid^{\delta/N-\sigma}$ as $\mid \xi \mid \to \infty$. If $\sigma > \delta/N$, we have that the corresponding

constant $C$, can be selected as 0 and $P_1 + iP_2^{1/N}$ is hypoelliptic. Conversely, any polynomial operator, hypoelliptic in $\mathcal{D}'$, can be constructed in this way.

Assume $P$ with $\sigma > 0$ and real. If $E$ is a fundamental solution to $P$, we have $\| Ef \|_{L^2} \leq C \| f \|_{L^2}$. Assume $f$ with first derivatives in $L^2$, since $P$ is real we have that $\overline{\delta} E = E\overline{\delta}$, why $E : H(\Omega) \to H(\Omega)$ for an open set $\Omega \subset \mathbf{R}^n$. As $P(D)u = f$ can be written $Ef = u$ in $L^2$, we conclude that $P$ is analytically hypoelliptic in $L^2$. Assume $P(D_x)$ an operator with constant coefficients and $I_E$ a parametrix. Then,

$$\overline{\delta} P I_E - P I_E \overline{\delta} = \big(\overline{\delta} P I_E - P\overline{\delta} I_E\big) + \big(P\overline{\delta} I_E - P I_E \overline{\delta}\big)$$

The first expression is trivially 0. If $u \in H \cap L^2(\Omega)$, $\overline{\delta} u = 0$, so $-\overline{\delta} P I_E(u) = P\big(\overline{\delta} I_E - I_E \overline{\delta}\big)$. Since $PI_E = I - \gamma$, $\gamma$ regularizing, we have

$$\overline{\delta}\gamma u = P\big(I_E\overline{\delta} - \overline{\delta} I_E\big)u \in C^\infty.$$

Thus, if $P$ is homogeneously hypoelliptic in $L^2$, $\Psi = I_E\overline{\delta} - \overline{\delta} I_E$ is regularizing. Finally, $\| \overline{\delta} I_E u \| \leq C \| \overline{\delta} u \|$ for a constant $C$, given that the set $\Omega$ is bounded. Thus $I_E : H(\Omega) \to H(\Omega)$ for bounded $\Omega$.

**Proposition 4.0.5.** *If $P$ is a constant coefficients operator, hypoelliptic in $L^2$, then $P$ is analytically (homogeneously) hypoelliptic in $L^2$. Conversely, if $P$ is analytically (homogeneously) hypoelliptic in $L^2$, $P$ is hypoelliptic in $L^2$.*

## 5. Some remarks on fundamental solutions

Assuming that $\Omega$ is an open set in $\mathbf{R}^{n+m}$, we let $\Omega_\varphi = \{(x,y) \,; \{(x,y)\} - \text{supp } (\delta_0 \otimes \varphi) \subset \Omega\}$, where $\delta_0 \otimes \delta_0 = \delta_0$ is the Dirac measure in $\mathbf{R}^{n+m}$ and $\varphi \in C_0^\infty(\mathbf{R}^m)$. We then know that the partial convolution $u *'' \varphi = u * (\delta_0 \otimes \varphi)$ is well defined in $\Omega_\varphi$, for $u \in \mathcal{D}'(\Omega)$. In the same way we can define $u *' \varphi$ in

$$\{(x,y); \{(x,y)\} - \text{supp } \varphi \times 0 \subset \Omega\},$$

for $\varphi \in C_0^\infty(\mathbf{R}^n)$. The partial convolution $u *'' v$, can also be defined for $u \in \mathcal{D}'$ and $v \in \mathcal{E}'$ in $\Omega_v$. We note that:

$$u \in H^{s,-N}_{loc}(\Omega) \Rightarrow u *'' \varphi \in H^{s,0}_{loc}(\Omega_\varphi) \tag{8}$$

(cf. [2]). For any partial differential operator $Q$ with constant coefficients, we have a fundamental solution $E_0$ in $\mathcal{D}'^F(\mathbf{R}^n)$, that is distributions of finite order (and a $E_0 \in B^{loc}_{\infty,\tilde{Q}}$, [8], sec. 10.2 ). Assume for instance $Q(D_y)E_0 = \delta_0$. We then have for $u \in \mathcal{E}'(\Omega)$,

$$Q(u *'' E_0) = u * (\delta_0 \otimes QE_0) = u.$$

For a hypoelliptic operator $M(D_x)$ with constant coefficients, we have a fundamental solution as above, now also with sing supp $F_0 = \{0\}$ and as before $M(u *' F_0) = u$ in $\Omega$. By multiplying $E_0$ with a $\chi \in C_0^\infty(\mathbf{R}^m)$, $\chi = 1$ in a neighborhood of a compact neighborhood of $(0,0)$, $K_y$, we have $\chi E_0 \in \mathcal{E}'$. Define a linear operator $E : \mathcal{E}' \to \mathcal{E}'$ by Ef $= f *'' \chi E_0$ for $f \in \mathcal{E}'(\Omega)$. Define $F : \mathcal{E}' \to \mathcal{E}'$ as $Fu = u *' \vartheta F_0$, $F_0$ corresponding to $M$ and $\vartheta = 1$ in a neighborhood of $K_x$ as above. Obviously, we have

$$E : H_W^{s,t'} \to H_W^{s,t}(Q), \quad F : H_W^{s,t'} \to H_W^{p_s,t'},$$

for some $t' > t$ and $EFf = f * (F_0 \otimes E_0)$ on $W$. Here $W$ is a compact set in $\Omega$.

For the operator $P(x^0, y^0, D_x)$, we can now construct a fundamental solution with singularity in $(x^0, y^0)$ as follows. Let $G_0$ be the fundamental solution, modified to

$\mathcal{E}'$, corresponding to

$$P(x^0, y^0, D_x) = \sum_j P_j(x^0, y^0, D_x)$$

on a compact neighborhood of $(x_0, y_0)$, $W$ and $Gf = f *' G_0$. If $f = \delta_{(x_0,y_0)}$ and $I$ is the trivial mapping, we get the fundamental solution to $P$ as $GI\delta_{(x_0,y_0)} = G_0 \otimes \delta_{y_0}$. It can also be shown that a linear mapping $F$ can be constructed for a formally hypoelliptic operator, which gives a fundamental solution to the variable coefficients operator $P(x, y, D_x)$, but we will use another method to produce this solution in chapter 12. We have the following lemma

**Lemma 5.0.1.** *There is a fundamental solution with singularity in $(x_0, y_0)$, to a constant coefficients differential operator on $\mathbf{R}^{n+m}$, independent of the variables in $\mathbf{R}^m$-space, $P(x_0, y_0, D_x)$, on the form $G_0 \otimes \delta_{y_0}$, where $G_0$ is a fundamental solution with singularity in $x_0$, to the operator, considered as an operator on $\mathbf{R}^n$.*

## 6. Realizations

We assume $\Omega$ open and connected and of dimension $n > 1$. We can define a linear mapping $\tilde{M}_0$ on $C_0^\infty(\Omega)$, as $\tilde{M}_0\varphi = M(D_x)\varphi$. This is a symmetric operator, densely defined in $L^2(\Omega)$. The minimal operator, $M_0$, or the strong extension, is defined as the closure of $\tilde{M}_0$ in $L^2(\Omega)$. This is a closed, linear, symmetric, densely defined mapping on $L^2(\Omega)$. The adjoint, $M_0^*$, is a closed, linear densely defined operator and $u \in \mathcal{D}_{M_0^*}$ if and only if in distribution sense $M_0^* u = M(D_x)u \in L^2(\Omega)$. $M_0^*$ is called the maximal operator or the weak extension. $M_0$ is self-adjoint if and only if $M_0^*$ is also symmetric.

For a self-adjoint extension, $\mathcal{A}$, of $M_0$ we must have $M_0 \subseteq \mathcal{A} \subseteq M_0^*$ and $\mathcal{A}$ is called a self-adjoint realization of $M(D_x)$ in $L^2(\Omega)$. The minimal- and maximal operators can also be defined for the operator $P(x, y, D_x)$ just as above. We will later on rationalize the calculations by comparing norms for realizations corresponding to different operators. For this we need:

**Lemma 6.0.1.** *For the operator $P$, FHE in $x$ of type $M$, we have that the domains of the respective closures in $H_K^{0,-N}$ of the operators coincide. That is*

$$\mathcal{D}(M\tilde{\otimes}I) = \mathcal{D}(P\tilde{\otimes}I)$$

*In particular ([24],Ch. 5) $\mathcal{D}((M \otimes I)_0) = \mathcal{D}((P \otimes I)_0)$. For Mizohata's representation $L$, we have that the domains of $\tilde{L}$ and $P\tilde{\otimes}I$ are partially coinciding and $\mathcal{D}(\widetilde{L}) \subset \mathcal{D}(P\tilde{\otimes}I)$.*

Proof: Assume for $\varphi_n \in C_c^\infty(K)$,where $C_c^\infty$ denotes $\{f \in C^\infty;\ \text{supp } f \subset K\}$ , a dense subset of $H_K^{s,t}$, $K$ is a compact set in $\Omega$, where $P$ is defined, $\varphi_n \to 0$ and $P(x, y, D_x)\varphi_n \to f$ in $H_K^{0,-N}$. If we can show that

$$\| M(D_x)\varphi \|_{0,-N} \leq C_1 \| \varphi \|_{H_K^{0,-N}(P)} \tag{9}$$

we see that $M(D_x)\varphi_n$ is convergent in $H_K^{0,-N}$ and since $M\tilde{\otimes}I$ is a closed operator, $M(D_x)\varphi \to 0$. If in addition

$$\| P(x, y, D_x)\varphi \|_{0,-N} \leq C_2 \| \varphi \|_{H_K^{0,-N}(M)} \tag{10}$$

and since we know that the right hand side tends to 0, we must have that $P(x, y, D_x)\varphi_n \to 0$ in $H_K^{0,-N}$, that is $f = 0$. The inequalities (9),(10) follow

immediately from what was said in section 3.1. The second statement also follows from section 3.1, assuming the rightmost norms finite

$$\parallel P(x,y,D_x)\varphi \parallel_{0,-N} \leq C_0' \parallel \varphi \parallel_{H_K^{0,-N}(M)} \leq C_1' \parallel \varphi \parallel_{H_K^{0,-N'}(L)}$$

and

$$\parallel L(x,y,D_x,D_y)\varphi \parallel_{0,-N'} \leq C_2' \parallel \varphi \parallel_{H_K^{0,-N}(M)} \leq C_3' \parallel \varphi \parallel_{H_K^{0,-N}(P)} . \Box$$

## 7. Some remarks on Schwartz kernels

Given $f \in C^\infty\ (X \times Y)$, $X, Y$ open sets in $\mathbf{R}^\nu$, we can define an integral operator on $\mathcal{D}\ (Y)$ according to

$$I_f(\psi)(x) = \int f(x,y)\psi(y)dy$$

which is continuous $\mathcal{D}\ (Y) \to C^\infty\ (X)$. This is a regularizing operator, it can be extended to a continuous linear operator $\mathcal{E}'\ (Y) \to C^\infty\ (X)$. More generally, we have Schwartz kernel theorem, that is given $K \in \mathcal{D}'\ (X \times Y)$, we can define a continuous, linear operator $\mathcal{D}\ (Y) \to \mathcal{D}'\ (X)$ according to

$$< I_K(\psi), \varphi > = < K, \varphi \otimes \psi > \qquad \psi \in \mathcal{D}(Y), \varphi \in \mathcal{D}(X) \tag{11}$$

and conversely, given the continuous, linear operator $I_K$, we have unique existence of a kernel $K \in \mathcal{D}'\ (X \times Y)$ such that (11) holds.
For a constant coefficients differential operator $Q$, let
$Q' = Q(D_x), Q'' = Q(D_y), \overline{Q''} = Q(-D_y)$. For test functions $f, g \in C_0^\infty(\mathbf{R}^\nu \times \mathbf{R}^\nu)$,

$$I_g(I_{Q'f}(\phi))(x) = I_g(Q'I_f(\phi))(x) = I_{\overline{Q''}g}(I_f(\phi))(x)$$

Assume $f \in C_0^\infty(\mathbf{R}^\nu \times \mathbf{R}^\nu)$ such that for constant coefficients operators, $\overline{Q'}f = Q''f$. Assume $P, Q, R$ such operators, then

v) $I_{\big[\big[P''f,Q''f\big],R''f\big]} = I_{\big[\big[f,P''f\big],Q''R''f\big]} = I_{\big[\big[f,f\big],P''Q''R''f\big]}$

Finally, we will have use for the concept of analytic functionals as in [13]. Assume $V$ a complex analytic variety. For analytic functionals defined on $V$, we have

$$I_F : H(V_y) \to H'(V_x) \qquad F \in H'(V_x \times V_y)$$

If $V$ is countable in the infinity, it can be shown that $H'(V)$ is a nuclear ($\mathcal{DS}$)-space (the dual topology to the ($\mathcal{FS}$)-topology, that is Frechet with Schwartz property). Iteration of the integral operators $I_F$ is, for our applications, defined by convolution. We note that ([13] Ch. 1, Proposition 1.1) *Any analytic functional defined on $V$, can be represented by a measure with compact support*
In the same way as in $\mathcal{D}'$, we can using nuclearity in $H'$, determine the kernel to the evaluation functional, as the parameter $x$ varies. For $\text{ Id }_x \in H'$, we have $< \text{ Id }_x, \psi > = \int \phi(z)\psi(z)dz = < \delta_x(\phi), \psi >$, with a $\phi \in C_0^\infty(\mathbf{R}^\nu)$ and $\psi \in H(\mathbf{C}^\nu)$. Let $\Delta \in H'(\mathbf{C}^\nu \times \mathbf{C}^\nu)$, be the kernel representing $\text{ Id }_x$. We note, that in a neighborhood of $x$, $V$, we have for this kernel,
$< I_\Delta^N(\varphi), \psi >_V = < I_{\big[\Delta,\Delta\big]_N}(\varphi), \psi >_V = 0$, with $\varphi, \psi \in H(\mathbf{C}^\nu)$, for some $N$ (cf. [4]).

## 8. The spectral kernel

According to the spectral theorem, any self-adjoint realization as in the previous section, can be represented as a spectral operator of scalar type:

$$\mathcal{A} = \int \lambda dE_\lambda \qquad \lambda \text{ real}$$

For the iterated operators, we also have realizations that are of scalar type:

$$\mathcal{A}_{L^r} = \int \lambda dE_r(\lambda) \qquad \text{where} \quad E_r(\lambda^r) = E(\lambda) \quad \text{, r is assumed odd}$$

Assuming the constant coefficients operator $L$ defined on $\mathbf{R}^\nu$, since $L$ is assumed formally self-adjoint, the realization is related to the symbol of the operator as $\mathcal{A}_L = \mathcal{F}L(\xi,\eta)\mathcal{F}^{-1}$. Let $\{E_\lambda\}$ be the spectral family corresponding to the realization $\mathcal{A}_L$ with spectrum $\sigma(\mathcal{A}_L) = \{L(\xi,\eta) \quad \xi,\eta \quad \text{real}\}$. We can assume the spectrum left semi-bounded. We then know that $E_\lambda$ is related to $\omega = \chi_{(-\infty,\lambda)} \circ L$, through $\mathcal{F}(E_\lambda u) = \omega\mathcal{F}u$ with $\mathcal{F}^{-1}\omega \in \mathcal{D}'$ ([19]).

For the type polynomial $M(\xi)$, we know that $M(\xi) \to \infty$ as $\mid \xi \mid \to \infty$, this means that the set $G_\lambda = \{\xi; M(\xi) < \lambda\}$ is bounded for every $\lambda$. Then, for the spectral kernel $e_\lambda(x_0,x) = \mathcal{F}^{-1}\chi_{G_\lambda}(x_0 - x) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ ([20]). We have

$$e_\lambda(x_0,x) = (2\pi)^{-n} \int_{M(\xi)<\lambda} e^{i(x_0-x)\cdot\xi} d\xi \tag{12}$$

Let $(x,y) = (x',x'') \in \mathbf{R}^\nu$, $\nu = \mathrm{n} + \mathrm{m}$. For the contact polynomial $L(\xi)$, that is the polynomial corresponding to the frozen operator $L(x_0, D_{x'}, D_{x''})$, this argument only holds for the hypoelliptic part of the polynomial. Consider instead

$$L(x_0,D_x) = \sum_j c_j(x_0)P_j(D_{x'})Q_j(D_{x''}),$$

where $P_j \sim M$ and $P_j, Q_j$ are constant coefficients, finite order operators. We can write $R(P,Q) = L$, where $R$ is a constant coefficients polynomial. Let $T_1 = P \otimes Id_1, T_2 = Id_2 \otimes Q$, where $Id_j, j = 1,2$ are identity operators in respective complex Banach space. It can be verified for the joint spectrum, $\sigma(T_1,T_2) = \sigma(T_1)\sigma(T_2) = \sigma(P)\sigma(Q)$. Further $T_1T_2 = T_2T_1$. Finally, $\sigma[R(T_1,T_2)] = R[\sigma(T_1,T_2)]$, ([23]).

Assume $E_{\lambda,\mu}$ the spectral resolution corresponding to the operator $L$ and consider the restriction to $\mathcal{D}(\mathbf{R}^\nu \times \mathbf{R}^\nu)$. Let $E_\lambda, E_\mu$ be the resolutions corresponding to $T_1$ and $T_2$, restricted to $\mathcal{D}$. According to what has been said above, for $v \in \mathcal{D}$, $\mathcal{F}\ (E_{\lambda,\mu}v) = w\mathcal{F}\ v$ with $w = w_\lambda w_\mu$ and $E_\lambda v = v *' \mathcal{F}^{-1}\ w_\lambda$ and $E_\mu v = v *'' \mathcal{F}^{-1}$ $w_\mu$. The tensor product of projections $E_L(J \times K) = E_{T_1}(J)E_{T_2}(K)$, is well defined for Baire sets $J, K$ in $\mathbf{R}$ ($Q$ is assumed to have real coefficients). Assume $E_\lambda$ has spectral kernel

$$t^1_\lambda(x,y) = e_{x',\lambda} \otimes \delta_{x''} = \frac{1}{(2\pi)^\nu} \int_{P(\xi')<\lambda} e^{i(x-y)\cdot\xi} d\xi$$

and $E_\mu$ with spectral kernel $t^2_\mu = \delta_{x'} \otimes e_{x'',\mu}$. Note that $e_{x',\lambda}(x',y') =$ $\mathcal{F}^{-1}\ w_\lambda(x'-y')$ and analogously for $e_{x'',\mu}$. We then have, for $f \in \mathcal{D}(\mathbf{R}^\nu_{\mathrm{y}})$

$$E_{\lambda,\mu}(f)(x) = I_{e_{x,\lambda,\mu}}(f)(x) = I_{t^1_\lambda}(I_{t^2_\mu}(f))(x) = I_{\left[t^1_\lambda, t^2_\mu\right]}(f)(x) = I_{e_{x',\lambda} \otimes e_{x'',\mu}}(f)(x)$$

We then know for the spectral function corresponding to the operator $P$ (defined as the type operator outside a compact set), $e_{x',\lambda} \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and for the spectral kernel corresponding to the operator $Q$ (defined as the identity operator outside the same compact set), $e_{x'',\mu} \in \mathcal{D}'(\mathbf{R}^m \times \mathbf{R}^m)$. Using that $E_{\lambda,\mu}$ is an orthogonal projection,

$$E^2_{\lambda,\mu} = I_{\left[e_{x,\lambda,\mu}, e_{x,\lambda,\mu}\right]} = I'^2_{e_{x',\lambda}} I''^2_{e_{x'',\mu}} = I'_{e_{x',\lambda}} I''_{e_{x'',\mu}} = E_{\lambda,\mu}$$

Particularly, if $\widetilde{e}_{x'',\mu} = e_{x'',\mu} *_x \varphi *_y \psi$, for test functions with support in a neighborhood of the origin in $\mathbf{R}^m$,

$$I_{\left[e_{x'',\mu} *_x \varphi, e_{x'',\mu} *_y \psi\right]} = I_{\widetilde{e}_{x'',\mu}}$$

and since $\widetilde{e}_{x'',\mu} \in C^\infty(\mathbf{R}^m \times \mathbf{R}^m)$, we can define $\widetilde{E}_{\lambda,\mu} = I_{e_{x',\lambda} \otimes \widetilde{e}_{x'',\mu}}(f)(x)$ for $f \in L^2(\mathbf{R}^\nu)$. This means that we have partial regularity, for the spectral kernel.

## 9. Topology. Notation and fundamentals

In what follows, the presentation is more or less a modification of the article (Nilsson [17]) and we refer to this work for proofs and arguments. Assume the operator $L(x, D_x)$ defined as partially formally hypoelliptic in a neighborhood of $x_0$, included in $K$, $K$ a compact subset in $\Omega \subset \mathbf{R}^\nu$ and as the type operator with constant coefficients outside $K$. We also assume, as has been mentioned earlier, that the type polynomial $M$ is real and such that $M(\xi') \geq 1$, for all $\xi' \in \mathbf{R}^n$ and for positive constants $C$ and $\varrho$, $M(\xi') \geq C \mid \xi' \mid^\varrho$, for all $\xi' \in \mathbf{R}^n$. According to these conditions, we get for this modified operator Re $L(x, \xi) \to \infty$ as $\xi' \to \infty$ while $\xi''$ bounded.

Next we will construct and estimate a fundamental solution to the variable coefficients operator $L(z, D_z) - \lambda$ and for this purpose we need an estimate of the distribution

$$\alpha_\lambda(x, z) = [P^x(D_{z'}) - L(z, D_z)]K_\lambda^+(x, z)$$

Here $K_\lambda^+(x, z)$ is a fundamental solution with singularity in $x$, corresponding to the modified operator $P(x, D_{z'}) - \lambda$, $\lambda$ is assumed large and negative. So $K_\lambda^+(x, \cdot) \in H_{loc}^{t,-N}(\mathbf{R}^\nu)$, can be written as $\chi_{x_0}(x)h_\lambda \otimes \delta_{x''}(x, z) +$ $(1 - \chi_{x_0}(x))t_\lambda \otimes \delta_0(x, z)$, where $\chi_{x_0} \in C^\infty(K)$ assumes the value 1 in a neighborhood of $x_0$, $t_\lambda$ is the fundamental solution to the type operator and $h_\lambda$ the fundamental solution to the operator $P(x, D_{z'}) - \lambda$ (section 5). With this modification $\alpha_\lambda(x, \cdot) \in \mathcal{E}'(\mathbf{R}^\nu)$.

Consider the norms

$$M_\alpha(u) = \int \mid \xi \mid^\alpha \mid \mathcal{F}u(\xi) \mid d\xi$$

over $\mathcal{S}'(\mathbf{R}^\nu)$, for any multi-index $\alpha$, where $\mid \xi \mid^\alpha = \sum_{\beta \leq \alpha} \mid \xi^\beta \mid$. These norms $M_\alpha$ define Banach spaces $\mathcal{B}_\alpha$ with the test functions $C_0^\infty$ as a dense subset. Also consider the operator norms

$$N^{\alpha,\beta}(L) = \sup_{0 \neq u \in \mathcal{B}_\alpha} \frac{M_\beta(Lu)}{M_\alpha(u)}$$

over linear mappings L from $\mathcal{B}_\alpha$ to $\mathcal{B}_\beta$

## 10. Complex translations in the polynomials

Now to the distribution $\alpha_\lambda(x, z) = \beta_\lambda(x, z) + r_\lambda(x, z)$, where

$$\beta_\lambda(x, z) = (P^x(D_{z'}) - P(z, D_{z'}))K_\lambda^+(x, z) \tag{13}$$

Since $(1 - \Delta'')^{-t}\delta_{x''}$ has Fourier transform in $L^1(\mathbf{R}^m)$, for $t > m$, we see that $\beta_\lambda *_x'' \mathcal{U}_{-2t}(x, z) \in (\mathcal{B}_0)_x$, where $\mathcal{U}_{-t}$ is according to proposition 3.2.1. For the last term, we know that $r_\lambda(x, z) = \sum_j P_j(z, D_{z'})Q_j(D_{z''})K_\lambda^+(x, z)$ weighted with $(1 - \Delta'')^{-t'}$, $t' > max_j\{deg(Q_j)\} + m$, is in $(\mathcal{B}_0)_x$. Further for $t' > max_j\{deg(Q_j)\}$, $\mathcal{F}_z\big(r_\lambda *_x'' \mathcal{U}_{-2t'}\big)$ is essentially bounded. The following

proposition and proof is close to [17] Lemma 8. Let
$\alpha_{\lambda,N,t}(x,y) = \alpha_\lambda *''_x \mathcal{U}_{\text{-}2N} *''_y \mathcal{U}_{\text{-}2t}(x{,}y)$ and analogously for $\beta_{\lambda,N,t}$.

**Proposition 10.0.1.** *There are positive numbers $c$ and $\kappa_0$ such that*

$$N^{\alpha,\alpha}(exp(\kappa \mid \lambda \mid^b (z'_j - x'_j))I_{\alpha_{\lambda,N,t}}(x,z)) = O(1) \mid \lambda \mid^{-c} \qquad \lambda \to -\infty \tag{14}$$

*for every $j$, $1 \leq j \leq n$, for every real $\kappa$ such that $\mid \kappa \mid \leq \kappa_0$, for every multi-index $\alpha$ and for some positive numbers $N$,$t$. The norms $N^{\alpha,\alpha}$ are taken with respect to $z$.*

**Remark** : Here $b$ is associated to the type polynomial according to

$$\mid D^\alpha M(\xi') \mid \leq C(1+ \mid M(\xi') \mid)^{1-b|\alpha|} \tag{15}$$

**Lemma 10.0.2.** *For $\beta_\lambda(x,z)$ in (13)*

$$N^{\alpha,\alpha}(I_{\beta_{\lambda,N,t}}(x,z)) = O(1) \mid \lambda \mid^{-c} \qquad \lambda \to -\infty$$

*with $N, t$ as in the proposition.*

Proof: We write $\beta_\lambda$ on the form

$$\beta_\lambda(x,z) = \sum_j \Big[b_j(x) - b_j(z)\Big] N_j(D_{z'})K_\lambda^+(x,z)$$

where $b_j$ are the $C^\infty$-coefficients to the operator $P(z, D_{z'})$ (the coefficients to $\beta_\lambda(x,z)$ vanish in $x = z$), $N_j$ a constant coefficient operator equivalent in $x'$ with the type operator and $K_\lambda^+(x,z) = h_\lambda \otimes \delta_{x''}$, the fundamental solution to the operator $P(x, D_{z'}) - \lambda$. Expansion in a Taylor series gives
$\beta_\lambda(x,z) = \sum_{\lambda,\mu} F_{\lambda,\mu}(x,z)$, where each term is on a form

$$F_\lambda(x,z) = \sum_{0<|\beta|<k} i^{|\beta|}(\beta!)^{-1}\Big(D^\beta b(z)\Big)(x-z)^\beta N(D_{z'})K_\lambda^+(x,z) + R_\lambda(x,z)$$

and

$$R_\lambda(x,z) = \frac{1}{(k-1)!}\Big(\int_0^1 (1-k)^{k-1}\frac{d^k}{dt^k}b(z+t(x-z))dt\Big)(x-z)^\beta N(D_{z'})K_\lambda^+(x,z)$$

We now study the two mappings

$$L_1 : u \to (D^\beta b)u \qquad L_2 : v \to const \int (x-z)^\beta N(D_{z'})K_\lambda^+(x,z)v(z)dz$$

and we aim to prove a weighted analogue to

$$N^{\alpha,\alpha}(L_2 \circ L_1) = O(1) \mid \lambda \mid^{-c} \qquad \lambda \to -\infty \tag{16}$$

$$N^{\alpha,\alpha}(I_{R_\lambda}) = O(1) \mid \lambda \mid^{-c} \qquad \lambda \to -\infty \tag{17}$$

Proof of (16):
The first mapping is immediate, since we already know the coefficients are in $\mathcal{B}_\alpha$ (or constant), which is a Banach algebra. So $N^{\alpha,\alpha}(L_1) < \infty$. For $L_2$, we set $K_\lambda^+(x,z) = H_\lambda(x,z) + \widetilde{H}_\lambda(x,z)$, where

$$H_\lambda(x,z) = \frac{1}{(2\pi)^\nu}\int \frac{e^{i(z-x)\cdot\xi}d\xi}{M(\xi') - \lambda}$$

We begin by proving the first claim for $H_\lambda(x,z)$. Let

$$p(x) = \int (x-z)^\beta N(D_{z'})H_\lambda(x,z)\varphi(z)dz$$

where $\varphi \in C_0^\infty(\mathbf{R}^\nu)$. Assume the partial convolution $\varphi *' f(x)$ defined by

$$\int \varphi(z', x'') f(x' - z') dz'$$

then

$$p(x) = \varphi^\vee *' \Big[(-x')^{\beta'} N(D_{z'}) \mathcal{F}_x^{-1'} \big[\frac{1}{M(\xi') - \lambda}\big]\Big] *'' \Big[(-x'')^{\beta''} \delta_0\Big]$$

Considering the partial Fourier transforms

$$\mathcal{F}' p(\xi', x'') = \big(\mathcal{F}'\varphi^\vee\big) D_{\xi'}^{\beta'} \Big[\frac{N(\xi')}{M(\xi') - \lambda}\Big] *'' \Big[(-x'')^{\beta''} \delta_0\Big]$$

$$\mathcal{F}'' p(x', \xi'') = (\mathcal{F}''\varphi^\vee) *' \Big[(-x')^{\beta'} N(D_{z'}) \mathcal{F}_x^{-1'} \big[\frac{1}{M(\xi') - \lambda}\big]\Big] \Big[D_{\xi''}^{\beta''} 1\Big]$$

we have

$$\mathcal{F}p(\xi) = \mathcal{F}\varphi(-\xi) \Big[D_{\xi'}^{\beta'} \big[\frac{N(\xi')}{M(\xi') - \lambda}\big]\Big]$$

and the outer bracket is estimated by Nilsson to be $O(1) \mid \lambda \mid^{-c}$ as $\lambda \to -\infty$. For the composed mapping $L_2 \circ L_1(\varphi)$, in order to make the norm $M_\alpha$ finite, we apply a weight operator in the "bad" variable. This weight is here (through duality), acting as an operator with constant coefficients, so $p(x)$ is replaced by

$$\big[D^\beta b\varphi\big]^\vee *' \Big[(-x')^{\beta'} N(D_{z'}) \mathcal{F}_x^{-1'} \big[\frac{1}{M(\xi') - \lambda}\big]\Big] *'' \Big[(1 - \Delta_{x''})^{-N} \delta_0\Big]$$

by choosing $N > \mid \alpha'' \mid$ and using that $I_{\big[\beta_\lambda *_x'' \mathcal{U}_{-2N}\big]}(\varphi)(x) = I_{\big[\beta_\lambda\big]}(\varphi) *_x'' \mathcal{U}_{-2N}(x)$, we have proved that $N^{\alpha,\alpha}(\big[(D^\beta b)(x - z)^\beta N(D_{z'}) H_\lambda\big] *'' \mathcal{U}_{-2N}(x,z)) = O(1) \mid \lambda \mid^{-c}$ as $\lambda \to -\infty$. Consider now $\widetilde{H}_\lambda(x, z)$, note that it is sufficient to consider the case when $x \in K$,

$$\widetilde{H}_\lambda(x, z) = \frac{1}{(2\pi)^\nu} \int \underbrace{\Big[\frac{1}{P^x(\xi') - \lambda} - \frac{1}{M(\xi') - \lambda}\Big]}_{G_\lambda(x,\xi')} e^{i(z-x)\cdot\xi} d\xi$$

With much the same arguments as in the previous case we get for

$$q(x) = \varphi^\vee *' \Big[(-x')^{\beta'} N(D_{z'}) \mathcal{F}_x^{-1'} G_\lambda(x, \xi')\Big] *'' \Big[(-x'')^{\beta''} \delta_0\Big]$$

that

$$\mathcal{F}q(\xi) = \mathcal{F}\varphi(-\xi) \Big[D^{\beta'} \big(N(\xi') G_\lambda(x, \xi')\big)\Big]$$

where the bracket according to Nilsson is $O(1) \mid \lambda \mid^{-c}$ as $\lambda \to -\infty$ so by applying the same weight as above we get

$$N^{\alpha,\alpha}(L_2 \circ L_1 *'' \mathcal{U}_{-2N}) = O(1) \mid \lambda \mid^{-c} \qquad \lambda \to -\infty$$

Proof of (17):
Now for the second claim, we can write $R_\lambda(x, z) = \sum_{\lambda, |\beta| = k} S_{\lambda,\beta}$ (a finite sum), where each term is on the form $S_{\lambda,\beta} = F_\beta(x, z) \widetilde{G}_{\lambda,\beta}(x, z)$. The conditions on the coefficients give that $F_\beta \in C^\infty(\mathbf{R}^\nu \times \mathbf{R}^\nu)$ and with bounded derivatives. Thus $R_\lambda(x, z) = 0$ for $\mid x \mid, \mid z \mid$ large. Using the tensor form $K_\lambda^+(x, z) = h_\lambda(x', z') \otimes \delta_{x''}(z'')$ we choose in this case to give a separate estimation of $\widetilde{G}_\lambda$ in the "good" variable. According to Nilsson

$$\mid D_{x'}^{\gamma'}((x' - z')^{\beta'} N(D_{z'}) h_\lambda(x', z')) \mid \leq C \mid \lambda \mid^{-c} (1 + \mid x' - z' \mid)^{-2(n+1)}$$

for $\mid \gamma' \mid \leq n + 1 + \mid \alpha \mid$. Using the inequality

$$(\Omega) \qquad \mid x - z \mid^\alpha \leq C \mid x \mid^\alpha \mid z \mid^\alpha$$

we get

$$| D_{x'}^{\gamma'} \mathcal{F}_x(R_\lambda(x', z') \otimes \delta_{x''}) | \leq \mathrm{C} | \lambda |^{-c} (1+ | x' |)^{-(n+1)}$$

Let $r(x) = \int R_\lambda(x,z)\varphi(z)dz$ then

$$| D_{x'}^{\gamma'} r(x) | \leq C | \lambda |^{-c} (1+ | x' |)^{-(n+1)} M_\alpha(\varphi)$$

where we have used that $\mathcal{B}_\alpha \subset \mathcal{B}_0$. Finally,

$$M_{\alpha'}(r) = \int \sum_{\gamma' \leq \alpha'} | \widehat{D^{\gamma'} r} | dx \leq C | \lambda |^{-c} M_\alpha(\varphi)$$

for $\lambda$ large and negative. If we apply the same weight operator as before we get $N^{\alpha,\alpha}(r * \mathcal{U}_{-2N}) = O(1) | \lambda |^{-c}$ as $\lambda \to -\infty$. This proves the second claim and we have proved the lemma.□

**Lemma 10.0.3.** *For $\beta_\lambda(x,z)$ in (13), we have*

$$N^{\alpha,\alpha}(\exp(\kappa | \lambda |^b (z'_j - x'_j)) I_{\beta_{\lambda,N,t}}(x,z)) = O(1) | \lambda |^{-c} \quad \lambda \to -\infty$$

*with $j, \kappa, \alpha$ and $t$ as in the proposition.*

Proof: follows immediately from the results in ([17]).

Proof of the proposition: We now consider the distribution (with compact support)

$$r_\lambda(x,y) = \sum_j b_j(z) R_j(D_{z'}) Q_j(D_{z''}) K_\lambda^+(x,z)$$

where $b_j$ are the $C^\infty$-coefficients corresponding to $L$, $R_j$ are constant coefficients operators strictly weaker than $M$ in $x'$, $Q_j$ are constant coefficients operators and $K_\lambda^+$ is the "parametrix" as before. We start by assuming that $\kappa = 0$. We make the same approach as in the proof of Lemma 10.0.2, noting that the Taylor series now contains a term $b_j(x)R_j(D_{z'})h_\lambda(x',z') \otimes Q_j(D_{z''})\delta_{x''}(z'')$. But we still have that $b(x) \in \mathcal{B}_\alpha$, so $N^{\alpha,\alpha}(L_1) < \infty$ (if $b(x)$ is constant there is nothing to prove).

For the term corresponding to $H_\lambda(x,z)$, we get with the same calculations as in the proof of (16)

$$\mathcal{F}p(\xi) = \mathcal{F}\varphi(-\xi)\Big[D^{\beta'}\big[\frac{R(\xi')}{M(\xi') - \lambda}\big]\Big]\Big[D^{\beta''}Q(\xi'')\Big]$$

for some test function $\varphi$. The last bracket can be handled by applying a weight operator of ("Sobolev")-order $-2t$, where $t > deg_{\mathbf{R}^m} Q + | \alpha'' |$. For the first bracket, Nilssons result still holds (according to the conditions on $R$, also for the term where $\beta = 0$), that is

$$\text{ess. sup } | D^{\beta'}\Big[\frac{R(\xi')}{M(\xi') - \lambda}\Big]\Big[\frac{D^{\beta''}Q(\xi'')}{(1+ | \xi'' |^2)^t}\Big] |= O(1) | \lambda |^{-c} \quad \lambda \to -\infty$$

For the second term corresponding to $\widetilde{H}_\lambda(x,z)$, we get the same results. So we conclude with this modification

$$N^{\alpha,\alpha}(L_2 \circ L_1 *_x'' \mathcal{U}_{-2N}) = O(1) | \lambda |^{-c} \qquad \lambda \to -\infty$$

Finally, for the analogue to (17), we estimate the "good" variable separately,

$$| D_{x'}^{\gamma'} \mathcal{F'}_x R_\lambda(x', z') | \leq C | \lambda |^{-c} (1+ | \xi' |^2)^{-(n+1)}$$

We replace $r(x)$ by

$$\tilde{r}(x) = \int R_\lambda(x',z') \otimes \big((1 - \Delta_{z''})^{-t} Q(D_{z''})\delta_{x''}(z'')\big)\varphi(z)dz$$

with $t > deg_{\mathbf{R}^m}Q + \mid \alpha'' \mid$ and by use of the inversion formula, we reach an estimate

$$\mid D_{x'}^{\gamma'}\tilde{r}(x) \mid \leq C' \mid \lambda \mid^{-c} (1+ \mid \xi' \mid^2)^{-(n+1)} M_\alpha(\varphi)$$

as before. We get the conclusion with an additional weight operator ($N >\mid \alpha'' \mid$) acting on the $x''$-variable, that is

$$N^{\alpha,\alpha}(\tilde{r} *'' \mathcal{U}_{-2N}) = O(1) \mid \lambda \mid^{-c} \qquad \lambda \to -\infty$$

Finally, the case $k \neq 0$ can be treated exactly as in the proof of Lemma 10.0.3, if we substitute $R$ for $N$ and $M$ for $P^x$ in the right side of $(\Theta)$, allowing $\beta' = 0$, since Lemma 1 in [17] can be applied without modifications on the strictly weaker operators.$\Box$

**Remark** :
Concerning the translations, we are content with the fact, that in the direction of the "bad" variable, translation is in general more difficult to handle, however we know that (Mizohata [15] Cor. 2) the mapping

$$\mathcal{E} \times H^{0,-N} \ni (c(z), T) \to c(z)T \in H^{0,-N}$$

is continuous with the additional assumption that $c(x) = 0$. Further
$\parallel c(z)T \parallel_{0,-N} \leq \epsilon \parallel T \parallel_{0,-N}$, where $\epsilon$ can be chosen arbitrarily small, as the support for $T$ tends to $\{x\}$. Thus

$$\parallel c_j(z'')Q(D_{z''})\varphi \parallel_{0,-N} \leq \epsilon \parallel Q(D_{z''})\varphi \parallel_{0,-N} \leq \epsilon C \parallel \varphi \parallel_{0,-N'}$$

We note that $\epsilon$ can be chosen as

$$\epsilon = \Big(\sup_{\mathrm{supp}\ \varphi} \mid c(z) \mid + d(\mathrm{supp}\ \varphi, x)\Big)$$

where $d$ denotes the greatest distance between $x$ and supp $\varphi$. We see that
$\mid c_j(z'') \mid = \mid e^{i\gamma(z_j''-x_j'')} - 1 \mid \leq 2$ for $\gamma$ real and that $\mid c_j(z'') \mid \to 0, \quad z_j'' \to x_j''$,
$1 \leq j \leq m$. Since the functions $(1+ \mid \xi'' \mid^2)^{-N}$ are very sensitive to complex translations, we must here assume that $N = 0$. This implies (study the mollifier $\varphi_k = \exp(ix \cdot \xi)\psi(x/k)/k^{n/2}$, as $k \to \infty$ for $\psi \in C_0^\infty$ and $\parallel \psi \parallel_{0,0} = 1$) for all $\xi'' \in \mathbf{R}^m$, for $\beta'' \neq 0$ and $d_Q = \deg_{\mathbf{R}^m} Q$

$$\mid D^{\beta''}Q(\xi'' - \gamma e_j'') - D^{\beta''}Q(\xi'') \mid \leq \epsilon C (1+ \mid \xi'' \mid^2)^{d_Q \mid \beta'' \mid}$$

Through the analytical properties of $Q(\xi'')$, we have that the inequalities can be extended to $\gamma = -i\kappa \mid \lambda \mid^b$, for $\kappa$ real with $\mid \kappa \mid \leq \kappa_0$ (although we have to assume $\lambda \neq \pm\infty$). Note that a better result can be found in [8] (Lemma 3.1.5, (1963) ).

However, since $\alpha_\lambda$ only involves the Dirac measure on the "bad" side, it is trivial that the proposition implies that, for every test function $\varphi \in C_0^\infty(\mathbf{R}^m)$

$$N^{\alpha,\alpha}(\exp(\kappa \mid \lambda \mid^b (z_j - x_j)) I_{\alpha_\lambda *_x'' \varphi)} = O(1) \mid \lambda \mid^{-c} \qquad \lambda \to -\infty$$

with $1 \leq j \leq \nu$, for all $\alpha$ and for $\kappa$ real with $\mid k \mid \leq k_0$.

We also need an estimate of $K_\lambda^+(x,z)$. We introduce the integrals

$$T_t^{2\alpha}(\lambda) = \int \frac{\xi^{2\alpha} d\xi}{(M(\xi') - \lambda)(1+ \mid \xi'' \mid^2)^t}$$

According to [17] Lemma 9, we have:

**Proposition 10.0.4.** *There is a positive constant $\kappa_0$ such that, when $1 \leq j \leq n$ and $\alpha$ is any multi-index, we have for all real numbers $\kappa$ with $| \kappa | \leq \kappa_0$ and for some positive $t$*

$$M_\alpha(exp(\kappa \mid \lambda \mid^b (y'_j - x'_j))K^+_\lambda *''_y \mathcal{U}_{-2t}(x,y)) = O(1)T^{2\alpha}_t(\lambda) \tag{18}$$

*$\lambda \to -\infty$, where the norm $M_\alpha$ is taken with respect to the variable $x$ and where the estimate is uniform with respect to $y \in \mathbf{R}^\nu$. Further $b$ is the positive number corresponding to $M$ as (15).*

*When $\beta \leq \alpha$, we also have, uniformly in $y \in \mathbf{R}^\nu$, for some positive $N$,*

$$M_\alpha(D^\beta_y K^+_\lambda *''_x \mathcal{U}_{-2N}(\cdot, y)) = O(1)T^{2\alpha}_N(\lambda) \tag{19}$$

*$\lambda \to -\infty$.*

## 11. The complex set of lineality

A constant coefficients operator $P(D)$, is called reduced, if for $\eta \in \mathbf{C}^n$,

$$P(\xi + t\eta) - P(\xi) = 0 \qquad \text{for all } \xi \text{ in } \mathbf{C}^n \text{ and all } t \in \mathbf{R} \quad \Rightarrow \eta = 0$$

We have use for the class $\mathcal{H}_\sigma$ of polynomials $P$, such that $| \xi |^\sigma \leq C \mid P(\xi) \mid$ for large $| \xi |$, for some constant $C$. Consider first, as before, complex translations in one variable, that is $\eta = i\eta_0 e''_j$, $1 \leq j \leq m$ , where $e''_j$ corresponds to a standard base in $\mathbf{C}^{\,m}$ and $\eta_0$ is some real number. The set of $\eta''$, such that $L(\xi + t\eta'') - L(\xi) = 0$, for our partially hypoelliptic operator $L = P_0 + R$, is a zero-dimensional analytic (algebraic) set, that can locally, be given as the zero-set corresponding to a polynomial. That is (disregarding points $\xi$ where $R(\xi) = 0$ and assuming $R$ dependent on all variables)

$$\Delta(L) = \{\eta''; \quad R(\xi', \xi'' + t\eta'') - R(\xi', \xi'') = 0 \ \forall t, \ \forall \xi\} = Z_\varphi$$

where $\varphi(\eta'')$ is a polynomial in a complex variable. The set $\Delta(L)$ will consist of a finite number of isolated points, that may cluster at the boundary of the local domain given for $\eta''$. We could say that the operator $L$ is reduced, with respect to $\varphi$. Further, for any truly reduced operator $Q$, we have that

$$\Delta(L) = \{\eta''; \quad Q(\xi', \xi'' + \varphi(t\eta'')) - Q(\xi', \xi'') = 0\}$$

Let's assume $Q \in \mathcal{H}_\sigma(\mathbf{R}^\nu)$ with $\sigma > 0$ and hypoelliptic in $L^2$. We can also assume, that a sufficiently large number of iterations of the operator $Q$, gives an operator hypoelliptic in $\mathcal{D}'$. Thus, if $1 \leq j \leq m$ is arbitrary, we have that $L^N$ is hypoelliptic, with respect to $\varphi$.

If we write the condition on reducedness,

$$\left[e^{i<x,t\eta''>} - 1\right]R(D) = 0 \quad \Rightarrow \eta'' = 0$$

and interpret the bracket as a hypoelliptic operator, dependent on a parameter $t$, on $L^2$, using the arguments in section 23, even though $R$, is not reduced, we will assuming $\left[e^{i<x,t\eta''>} - 1\right]R(D) = 0$, get

$$\left[e^{i<x,t\eta''>} - 1\right]^N R^N(D) = 0$$

for some positive integer $N$. This means particularly that

$$R^N(\xi', \xi'' + t\eta''_1) \pm \ldots \pm R^N(\xi', \xi'' + t\eta''_N) = R^N(\xi', \xi'')$$

The Fredholm theory gives, that the null-space to the translation operator, is stable from some iteration index $N$, so if $R^N$ is reduced, we have $\big[e^{i<x,t\eta''>} - 1\big]R^N(D) = 0 \Rightarrow \eta'' = 0$.

Assume now,

$$(1 + c \mid \zeta \mid)^k \mid \varphi_\lambda(\zeta) \mid^2 = Ch(\zeta)$$

for a holomorphic function $h \in \mathcal{O}(\ U')$,$U \subset U'$ an open set. We can assume $\mid h \mid \leq 1$ and the domain $U$ bounded. Let $g(\zeta) = Ch(\zeta) - (1 + \mid \zeta \mid)^k \mid \varphi_\lambda(\zeta) \mid^2$, be the holomorphic function defining $U$. As $\text{ord}_\zeta h \leq \text{ord}_\zeta g$, for every $\zeta \in U$, we have that $g$ is less than a constant times $h$ in modulus. Thus for m $= 1$, by Rouché's theorem, $g$ is a polynomial. In higher dimension, we can at least say that the restriction of $g$ to any complex line, is a polynomial. Let's define a real set

$$\Delta_{\mathbf{C}} = \{\eta \in \mathbf{R}^m; \quad R(\xi + it\eta) - R(\xi) = 0 \quad \xi \in \mathbf{R}^\nu, \quad t \in \mathbf{R}\}$$

**Lemma 11.1.** *If $R_\lambda$ is reduced for complex lineality $\Delta_{\mathbf{C}}$ (with respect to one dimensional translations), then $(1 + \mid \xi \mid)^c \leq C \mid R_\lambda(\xi) \mid$, for positive constants $c, C$ and $\xi \in \mathbf{R}^\nu$.*

Proof: ( sketch of a proof ) $R_\lambda$ can, as an entire function, be developed as

$$R(\xi + it\eta) - R(\xi) = \sum_\alpha \big[R_\alpha(it\eta) - R_\alpha(0)\big]\xi^\alpha = \sum_\alpha F_\alpha(it\eta)\xi^\alpha \tag{20}$$

We can without loss of generality, assume the following argument takes place outside the complex set of zero's to the polynomial $R$. If $\eta \in \Delta_{\mathbf{C}}$, $i\eta$ is a growth vector for $R$, ord ${}_0F_\alpha = +\infty$ and we can show $\mid F_\alpha(it\eta) \mid / \mid t\eta \mid^r \leq C_{r,t}$, for every positive real $r$, for every multiorder $\alpha$ and for $t$ close to 0. Assume $\Delta_{\mathbf{C}} = \{0\}$, it then follows ([3],Ch.1, section 5) that for $\eta \neq 0$

$$\mid F_\alpha(it\eta) \mid / \mid t\eta \mid^r > C_{t,\alpha} \quad \text{for some real } r \tag{21}$$

for $t$ close to 0. Further $\mid R_\alpha(it\eta) \mid / \mid t\eta \mid^r > C_\alpha$, for $t$ close to 0. This means, that

$$\mid \xi \mid^v C_{t,\alpha,\eta} \leq \sum_{\beta \leq \alpha, \mid\alpha\mid = v} \mid R_\alpha(it\eta)\xi^\beta \mid$$

Thus, for $\eta \notin \Delta_{\mathbf{C}} = \{0\}$, particularly, $\mid \xi \mid^v \leq C \mid R_\lambda(\xi) \mid$, for some $\lambda$. Note that $v$ can be chosen as the least $v$ for which $F_\alpha \neq 0$, $\mid \alpha \mid = v$.$\square$

For $h(\zeta) = 0$, since $g^t$ is reduced, for some positive integer $t$, there is a positive $\sigma$, such that $\mid \xi \mid^\sigma \leq \mid g(\xi) \mid$, for large $\xi \in \mathbf{R}^m$. Thus, $\mid \xi \mid^{t\sigma - k''} \leq C \mid \varphi_\lambda(\xi) \mid^{2t}$, for some positive $k''$. For $\mid h(\zeta) \mid > 0$, the inequality for $h$, is immediate. Assuming the defining polynomial is self-adjoint, we conclude that $\varphi_\lambda^t \in \mathcal{H}_{\frac{1}{2}(t\sigma - k'')}(\mathbf{R}^m)$. This means, for a sufficient number of iterations, $\sigma > k''/t$ and the polynomial is reduced. The constant $k'' < 1$, so there is a positive integer $N$ (possibly smaller than $t$), such that $\sigma > 1/N$ and $\varphi_\lambda^N$ is hypoelliptic in $L^2$. Note that this hypoellipticity is not dependent on $\lambda$.

Obviously, if $\mid \xi \mid^\sigma \leq C \mid \varphi_\lambda(\xi) \mid$, for some positive $\sigma$, then the same must hold for $R_\lambda$. Since, if $\eta \in \Delta_{\mathbf{C}}$, the $R_\alpha$'s are constants and if $\eta \notin \Delta_{\mathbf{C}}$, the result follows analogously to the second part of the proof of the previous lemma. Note that for a hyperbolic polynomial, this implication is not true. Since the set of lineality for

such an operator is determined by a homogeneous polynomial $Q$, if $\varphi_\lambda$ is reduced, we get something like $\mid \xi \mid^{-\sigma} \leq C \mid Q(\xi) \mid$, for a positive number $\sigma$.

Assume now, $\text{Re } P$ partially hypoelliptic. This is a self-adjoint operator, which means that there is a $N_0$ such that $(\text{ Re } P)^N$ is hypoelliptic for $N \geq N_0$. The same holds for the imaginary part. Assume $\text{Im } P \prec \text{ Re } P$, then $i(\text{ Im } P)^{N'} + (\text{ Re } P)^{N'} \prec P^{N'}$ and we can conclude that $P^{N'}$ is hypoelliptic. Thus the requirement on self-adjointness in the above argument is not essential in the proof of

**Proposition 11.2.** *If $P(D)$ is a constant coefficients, (self-adjoint), partially hypoelliptic operator, there is an iteration index $N_0$, such that $P(D)^N$ is hypoelliptic in $\mathcal{D}'$, for all $N \geq N_0$.*

Let $e^{i<x,\varrho>}$ denote the translation operator in $\| \cdot \|$, for $\varrho \in \mathbf{R}^n$, we then have a remainder operator $R(D)$ with constant coefficients dependent on the coefficients for $P(D)$ and on $\varrho$, such that

$$e^{i<x,\varrho>}R(D) = e^{i<x,\varrho>}P(D) - P(D)e^{i<x,\varrho>}$$

Further,

$$\big[e^{i<x,\varrho>} - 1\big]P(D) = e^{i<x,\varrho>}R(D) + P(D)\big[e^{i<x,\varrho>} - 1\big]$$

and we can prove, that there is a $\lambda_0$ large, such that

$$\| e^{i<x_j,|\lambda|>}R(D)u \| \leq C_\lambda \big( \| P(D)u \| + \| u \| \big) \qquad \mid \lambda \mid > \lambda_0 \quad u \in H_K^{0,0} \tag{22}$$

where the constant can be chosen, so that $C_\lambda \to 0$ as $\mid \lambda \mid \to \infty$. For a reduced operator, the distance function grows like $\mid \xi \mid$ and we have that $R \prec\prec P$, where the constant may depend on $\lambda$. For a non-reduced operator, we note that we may assume $\mid \xi \mid^q \leq \mid \lambda \mid$, as $\mid \lambda \mid \to \infty$, since with the opposite condition, the result is trivial. More precisely, assume $V_\lambda = \{\xi; \quad \mid \lambda \mid \leq \mid \xi \mid^q\}$. These sets will become insignificant as $\mid \lambda \mid \to \infty$, however we prove that $R \prec\prec P$ on this set, as $\mid \lambda \mid \to \infty$. The inequality $\mid \xi \mid^{2q} \leq \mid \lambda \mid d(\xi, \Delta_{\mathbf{C}})$ gives that $\mid \lambda \mid \leq Cd(\xi, \Delta_{\mathbf{C}})$ as $\mid \lambda \mid \to \infty$. Further $\mid \lambda \mid^\sigma \mid \xi \mid^\sigma \leq Cd(\xi, \Delta_{\mathbf{C}})$, so

$$\frac{\mid R(\xi) \mid}{1 + \mid P(\xi) \mid} \leq C \frac{1}{\mid \lambda \mid^\sigma} \to 0 \ \text{ as } \ \mid \lambda \mid \to \infty$$

This gives, with the first condition, for large $\mid \xi \mid$,
$\frac{1}{c} \mid \xi \mid^{\sigma - q} \mid R(\xi + \mid \lambda \mid e_j) \mid \leq \mid \lambda \mid + \mid R(\xi) \mid \mid \xi \mid^{q-\sigma}$, with $q$ according to ([21],Ch.II,ğ3,Prop.2) and $\sigma$ corresponds to the distance function to the lineality (one dimensional), for the operator $R$. The result follows, for $\sigma > q$.

## 12. The Levi parametrix method

The method used to construct a fundamental solution $g_\lambda(x,y)$ to the operator $L(y, D_y) - \lambda$, when $\lambda$ large and negative, is a modified version of Levi's parametrix method [10]. Assume $K_\lambda^+(x,z)$ a fundamental solution in $\mathcal{S}'$ to the operator with constant coefficients $P_\lambda^x(D_{z'}) = P(x, D_{z'}) - \lambda$, $P^x$ hypoelliptic in $z'$, that is the operator $P(z, D_{z'})$ in section 2 (1) frozen in $x$. This fundamental solution was discussed in section 5,

$$K_\lambda^+(x,z) = \begin{cases} h_\lambda(x', z') \otimes \delta_{x''}(z'') & x \in K \\ t_\lambda(x', z') \otimes \delta_{x''}(z'') & x \notin K \end{cases}$$

where $t_\lambda$ is the fundamental solution to the type operator regarded as an operator on $\mathbf{R}^n$. Note that $t_\lambda(x',\cdot), h_\lambda(x',\cdot)$ in $C^\infty(\mathbf{R}^n\setminus x')$. Let

$$g_\lambda(x,z) = K_\lambda^+(x,z) + \Big[u_\lambda, K_\lambda^+\Big](x,z) \tag{23}$$

where the bracket stands for $\Big[f,g\Big](x,z) = \int f(x,y)g(y,z)dy$. Assuming $u_\lambda$ can be constructed with certain given regularity properties, $g_\lambda(x,z)$ will be the fundamental solution to the operator $L_\lambda(z,D_z) = L(z,D_z) - \lambda$, that is

$$\overline{v(x)} = \int \overline{L_\lambda(z,D_z)[v(z)]} g_\lambda(x,z)dz \tag{24}$$

Using that $K_\lambda^+(x,y)$ is a fundamental solution to $P_\lambda^x(D_{z'})$ and assuming that $v \in \mathcal{S}$ (although it would be sufficient to assume $v \in \mathcal{E}$)

$$\int \overline{L_\lambda(z,D_z)[v(z)]} K_\lambda^+(x,z)dz = \int (\overline{L_\lambda(z,D_z)} - \overline{P_\lambda^x(D_{z'})})\overline{[v(z)]} K_\lambda^+(x,z)dz + \tag{25}$$

$$\int \overline{P_\lambda^x(D_{z'})[v(z)]} K_\lambda^+(x,z)dz = -\int \overline{v(z)} \alpha_\lambda(x,z)dz + \overline{v(x)}$$

where

$$\alpha_\lambda(x,z) = (P_\lambda^x(D_{z'}) - L_\lambda(z,D_z))K_\lambda^+(x,z) = \delta_x(z) - L_\lambda(z,D_z)K_\lambda^+(x,z)$$

Note that $K_\lambda^+$ acts as a "parametrix" also to the perturbed operator. We have for $x' \neq z'$, $-\alpha_\lambda(x,\cdot) \in C^\infty$ in $z'$ and for $\mid \beta' \mid < M$, ($M < \varrho - n$, $\varrho$ according to section 3.2) $D_{x'}^{\beta'} \alpha_\lambda(\cdot,z) \in C^0(\mathbf{R}^n)$. Using the expression (23) in (24) and the resulting equality in (25) and after reversing the order of integration in one of the integrals, we get

$$0 = \int \overline{v(z)} \Big( u_\lambda(x,z) - \alpha_\lambda(x,z) - \Big[u_\lambda, \alpha_\lambda\Big](x,z)\Big) dz \tag{26}$$

The bracket $\Big[f,g\Big](x,y)$, for $f,g \in \mathcal{B}_0$, can be regarded as a kernel to an operator $\mathrm{K} \in \mathcal{L}(\mathcal{B}_0)$, (here $\mathcal{B}_0$ is the space of tempered distributions with locally summable Fourier transforms), such that $K(w) = \int \Big[f,g\Big](x,z)w(z,y)dz$. Also, the kernel itself can be regarded as an operator in $\mathcal{L}(\mathcal{B}_0)$. In order to construct $g_\lambda$, we need to prove that, after modification, $\alpha_\lambda$ is bounded as an operator on $\mathcal{B}_\alpha$.

12.1. **The remainder $\alpha_\lambda$ weighted is in $\mathcal{B}_\alpha$.** A closer study of $\alpha_\lambda(x,z)$ gives first that $K_\lambda^+(x,\cdot)$ has partial Fourier transforms in $L^1_{loc}(\mathbf{R}^m)$ and $L^1(\mathbf{R}^n)$ respectively. Further the coefficients of

$$L_\lambda(z,D_z) = P_\lambda(z,D_{z'}) + R(z,D_z)$$

according to (1), have derivatives with compact support, which means that if they are not constant, they are in $\mathcal{B}_\alpha$, so $\mathcal{F}_x(P_\lambda(z,D_{z'})K_\lambda^+(x,z))(\xi) \in L^1_{loc}$ and partially in $L^1(\mathbf{R}^n)$. For the part of $\alpha_\lambda(x,z)$ involving derivatives in the "bad" variables, we get that

$$R(z,D_z)K_\lambda^+(x,z) = \sum_j P_j(z,D_{z'})Q_j(D_{z''})K_\lambda^+(x,z) =$$

$$\sum_{j,l} c_{j,l}(z)\Big(N_{j,l}(D_{z'})h_\lambda(x',z') \otimes Q_j(D_{z''})\delta_{x''}(z'')\Big) \tag{27}$$

so the conditions, $P_j \prec\prec_{z'} M$ and the coefficients, if not constant in $\mathcal{B}_\alpha$, give that the Fourier transform acting on $z$ and $RK_\lambda^+$, is in $L^1_{loc}(\mathbf{R}^\nu)$ and partially in $L^1(\mathbf{R}^n)$.

For the parameter $x$, we have that

$$t_\lambda \otimes \delta_{x''}(x,z) = \Big[\frac{1}{(2\pi)^n}\int \frac{\exp\Big[i(z'-x')\cdot\xi'\Big]d\xi'}{M(\xi')-\lambda}\Big] \otimes \delta_{x''}(z'')$$

so that $\mathcal{F}'_z t_\lambda(x',z') \in L^1(\mathbf{R}^n)$. The same results follow for $h_\lambda(x',z')$, from the conditions on the operator $P^x$. So $K_\lambda^+ *''_x \mathcal{U}_{-2t}$, $(LK_\lambda^+) *''_x \mathcal{U}_{-2t} \in \mathcal{B}_\alpha$, for some $t$.

**Remark** : It would be possible to construct, for the contact operators, a fundamental solution in the space of tempered distributions with Fourier transform in $L^1_{loc}(\mathbf{R}^\nu)$. We will however prefer a construction, in the space of analytic functionals.

Let $F_\lambda(x,z)$ be the expression within the parenthesis in (26), so that $\int \overline{v(z)} F_\lambda(x,z)dz = 0$, for all test functions $v \in \mathcal{O}$. For a fixed $x$, we then have $F_\lambda = 0$. This means that the problem of finding the fundamental solution $g_\lambda \in H'$ (both variables), is reduced to finding $u_\lambda$ such that in $H'(\mathbf{C}^\nu_z)$

(28) $$\alpha_\lambda(x,z) = u_\lambda(x,z) - \Big[u_\lambda, \alpha_\lambda\Big](x,z)$$

and such that in $H'(\mathbf{C}^\nu_x)$

$$\alpha_\lambda(x,z) = u_\lambda(x,z) - \Big[\alpha_\lambda, u_\lambda\Big](x,z)$$

12.2. **The remainder $\widetilde{\alpha}_\lambda$ is in $\mathcal{B}_\alpha$.** Using a partial mollifier, we see that $\Big[f,g\Big]$ is a continuous operator in both variables, on the space of tempered distributions with Fourier transforms in $L^1(\mathbf{R}^n)$ and $L^1_{loc}(\mathbf{R}^m)$ respectively, assuming $f,g$ have the same properties. The distribution $\alpha_{\lambda,N,t} = \alpha_\lambda *''_x \mathcal{U}_{-2N} *''_y \mathcal{U}_{-2t}(x,y)$, for $t > \max \deg Q_j + m + \mid \alpha'' \mid$ and $N > m$, is from the construction a linear, closed operator on the space $\mathcal{B}_\alpha$. This implies continuity by the closed graph theorem. We will now use the double partial regularization meaning

$$\alpha_\lambda *''_x \varphi *''_y \psi(x,y) = \iint \alpha_\lambda(x', x''-z'', y', y''-w'')\varphi(z'')\psi(w'')dz''dw''$$

The continuity implies that $\widetilde{\alpha}_\lambda = \alpha_\lambda *''_x \varphi *''_y \psi$ is in $\mathcal{L}(\mathcal{B}_\alpha)$ (and using the previous paragraph, it is also in $\mathcal{B}_\alpha$), since for $\varphi \in C_0^\infty(\mathbf{R}^m)$, we have $(1+ \mid \xi'' \mid^2)^N \widehat{\varphi} \in L^\infty$ (we denote the corresponding norm $\parallel \cdot \parallel_{\infty,N}$) for every integer $N$. Thus

$$M_\alpha(\alpha_\lambda *''_x \varphi *''_y \mathcal{U}_{-2t}) \leq \parallel \delta_0 \otimes \varphi \parallel_{\infty,N} M_\alpha((1-\Delta_{x''})^{-N}\alpha_\lambda *''_y \mathcal{U}_{-2t})$$

In the same way

$$M_\alpha(\alpha_\lambda *''_x \mathcal{U}_{-2N} *''_y \psi) \leq \parallel \delta_0 \otimes \psi \parallel_{\infty,t} M_\alpha((1-\Delta_{y''})^{-t}\alpha_\lambda *''_x \mathcal{U}_{-2N})$$

Particularly $\widetilde{\alpha}_\lambda \in \mathcal{B}_\alpha$ in both variables.

12.3. **Convergence for the series $u_\lambda$ in $\mathcal{E}'(\mathbf{R}^\nu \times \mathbf{R}^\nu)$.** If in the coefficients for $L$, the variable $y''$ is regarded as a parameter, we can write $\alpha_\lambda(x,y)$ on the form

$$\sum_j \Big(P_j^x(D_{y'}) - P_{j,(y'')}(y', D_{y'})\Big)h_\lambda(x',y') \otimes Q_j(D_{y''})\delta_{x''}(y'') =$$

$$\sum_j p_{\lambda,j,(y'')}(x',y') \otimes q_j(x'',y'')$$

Proposition 10.0.1 implies an estimate of the regularization, for $\varphi, \psi \in C_0^\infty(\mathbf{R}^m)$ $\alpha_{\lambda,(\sim)}(x,y) = \sum_j \big(p_{\lambda,j,(y'')} \otimes q_j *_x'' \varphi *_y'' \psi\big)(x,y)$, $N^{\alpha,\alpha}(I_{\alpha_{\lambda,(\sim)}}) = O(1) \mid \lambda \mid^{-c}$ as $\lambda \to -\infty$ (and it is not difficult to establish the same estimate for the norms taken with respect to $x$). For $\lambda$ sufficiently large and negative, this operator norm is $< 1$. This makes it possible to exclude the parameter $y''$ from the calculations, since according to the conditions on the coefficients, $\mid c_j(z',y'') - c_j(z',z'') \mid \leq C_1$ for $y'', z'' \in K$ and $\leq C_2$ for $z'' \notin K$, where we have constant coefficients. That is, for $\lambda$ sufficiently large,

$$M_\alpha\Big(\int p_{\lambda,j,(y'')}(x',z') \otimes \widetilde{q}_j(x'',z'')\varphi(z)dz\Big) \leq$$

$$M_\alpha\Big((\int_K + \int_{\mathbf{R}^\nu \setminus K})(p_{\lambda,j,(y'')} - p_{\lambda,j,(z'')})(x',z') \otimes \widetilde{q}_j(x'',z'')\varphi(z)dz\Big) +$$

$$M_\alpha\Big(\int p_{\lambda,j,(z'')}(x',z') \otimes \widetilde{q}_j(x'',z'')\varphi(z)dz\Big) < M_\alpha(\varphi)$$

We will use the same symbol $\alpha_{\lambda,(\sim)}$ for the case where the parameter $y''$ is "frozen" or excluded from the calculations. Note that, this means that the requirement that the operator $L_\lambda$ is formally partially self adjoint, is not necessary in the estimates of the fundamental solution, that we shall give. Let $u_{\lambda,(\sim)}$ denote the solution to the equation (28) corresponding to $\alpha_{\lambda,(\sim)}$. This is an integral equation of Fredholm type and a theorem from the theory of Fredholm operators on Banach spaces ([9] Lemma 2.5.4) gives existence of the solution $u_{\lambda,(\sim)}$, not dependent on the multi-index, in $\mathcal{L}(\mathcal{B}_\alpha)$ that is, for $\lambda$ sufficiently large and negative

$$u_{\lambda,(\sim)} = \alpha_{\lambda,(\sim)} + \Big[\alpha_{\lambda,(\sim)}, \alpha_{\lambda,(\sim)}\Big] + \Big[\alpha_{\lambda,(\sim)}, \alpha_{\lambda,(\sim)}\Big]_2 + \dots \tag{29}$$

where $\Big[f,f\Big]_2$ stands for $\Big[\Big[f,f\Big],f\Big]$ with convergence in $\mathcal{L}(\mathcal{B}_\alpha)$. Note that for $I_{u_{\lambda,(\sim)}} : \mathcal{B}_\alpha(\mathbf{R}^\nu_{\mathrm{y}}) \to \mathcal{B}_\alpha(\mathbf{R}^\nu_{\mathrm{x}})$, when $x$ is assumed fixed, we get that $I_{u_{\lambda,(\sim)}} \in \mathcal{B}_\alpha^{'}(\mathbf{R}^\nu_{\mathrm{y}}) \subset \mathcal{D}'(\mathbf{R}^\nu_{\mathrm{y}})$ and as usual, we identify $I_{u_{\lambda,(\sim)}}$ with the kernel $u_{\lambda,(\sim)}$. If we let $\widetilde{q}_j$ be the regularized elements, that is $q_j *_x'' \varphi *_y'' \psi$, the series (29) becomes

$$\sum_j p_{\lambda,j,(y'')} \otimes \widetilde{q}_j + \sum_{j,k} \Big[p_{\lambda,j,(y'')}, p_{\lambda,k,(y'')}\Big]' \Big[\widetilde{q}_j, \widetilde{q}_k\Big]'' +$$

$$\sum_{j,k,l} \Big[\Big[p_{\lambda,j,(y'')}, p_{\lambda,k,(y'')}\Big], p_{\lambda,l,(y'')}\Big]' \Big[\Big[\widetilde{q}_j, \widetilde{q}_k\Big]'', \widetilde{q}_l\Big]'' + ..$$

where the brackets are taken over $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. Let $I$ be the index set $\{i_0, i_1, \cdots, i_N, \cdots\}$, where every index assumes values in $\{1, \cdots, r\}$. For the brackets over $\mathbf{R}^n$, let for instance $\Big[p_{\lambda,i_0,(y'')}, p_{\lambda,i_2,(y'')}\Big]_2 =$ $\Big[\Big[p_{\lambda,i_0,(y'')}, p_{\lambda,i_1,(y'')}\Big], p_{\lambda,i_2,(y'')}\Big]$. The remainder, in the series (29) corresponding to $u_{\lambda,(\sim)}$ is then on the form

$$\sum_{|I|=N+1}^{\infty} \sum_I C_{i_0} \dots C_{i_{|I|}} \Big[p_{\lambda,i_0,(y'')}, p_{\lambda,i_{|I|+1},(y'')}\Big]'_{|I|+1}(x',y')\varphi \otimes \psi_{i_{|I|+1}}(x'',y'') \tag{30}$$

where $C_i = \int \psi_i(z'')\varphi(z'')dz''$ and where $\psi_i(z'') = Q_i(D_{z''})\psi(z'')$. Note that for every index, $\psi_i$ involves an application of an operator of finite order. The product of constants can be estimated through the estimated behavior of the "good" side, for $\lambda$ sufficiently large and negative. That is, we have $\mid R_{\lambda,(\sim)} \mid \leq const.e^{C} e^{-\kappa |\lambda|^b}$, where $C_{i_j} < C$ for every $i_j$. Note that if the test functions are assumed to have

support in a neighborhood of 0, then the convolution is defined, for $y''$ in a neighborhood of $x''$. This is sufficient for the study of the fundamental solution.

We claim that we have convergence $u_{\lambda,(\sim)}(x,\cdot) \rightarrow u_\lambda(x,\cdot)$ in $\mathcal{E}'$, as $\varphi \otimes \psi \rightarrow \delta_{x''}$. Using the equation following (28) and the parallel argument, the claimed convergence can be established in both variables separately. Let $u_{\lambda,N,(\sim)}$ be the $N+1$ first terms in (29), then immediately

$$u_{\lambda,N,(\sim)} - \alpha_{\lambda,(\sim)} = \Big[\alpha_{\lambda,(\sim)}, \alpha_{\lambda,(\sim)}\Big]_1 + \ldots + \Big[\alpha_{\lambda,(\sim)}, \alpha_{\lambda,(\sim)}\Big]_N = \Big[u_{\lambda,N-1,(\sim)}, \alpha_{\lambda,(\sim)}\Big] \tag{31}$$

We have established convergence as $N \rightarrow \infty$, for the left side in (31). According to ([17], Lemma 10), the "good" part in $u_{\lambda,N,(\sim)}$, that is the '-bracket in (30) (with the sum taken from 1 to $N$) is in $C^0(\mathbf{R}^n \times \mathbf{R}^n)$. Noting that these partial sums have compact support, we get that $u_{\lambda,N}(x,\cdot) \in \mathcal{E}'(\mathbf{R}^\nu)$. In section 12.5, we argue that this means that $u_\lambda(x,\cdot)$ is a linear form and a distribution in $\mathcal{E}'(\mathbf{R}^\nu)$ (Banach-Steinhaus theorem).

12.4. **Regularity properties for the series $u_\lambda$.** In this section we argue that even though $u_\lambda$ will contain differential operators of infinite order, it is for a study of the regularity properties, sufficient to study a finite number of terms, in the development of $u_\lambda$, that is the action of finite order differential operators. We start by noting the following trivial proposition.

*Assume $K \in \mathcal{E}'(\mathbf{R} \times \mathbf{R})$ the kernel to a continuous integral operator, $I_K : \mathcal{E}' \rightarrow \mathcal{E}'$ (or $B_\alpha \rightarrow B_\alpha$), such that $\mid \xi \mid^\sigma \leq \mid \mathcal{F}_y \, K(x,y) \mid$, $\quad \mid \xi \mid$ large, for $\sigma > 0$. Then $M_\alpha(u) \leq M_0(I_K(u))$, $\mid \alpha \mid \leq \sigma/2$, for $u$ a distribution, such that the right side is finite.*

The proof is immediate. Thus, such an operator is hypoelliptic on $\mathcal{E}'$ (or $B_\alpha$). We could choose $K(x,y) = P(D)\delta_x(y)$, for a reduced differential operator with constant coefficients. Further, we have the following proposition.

*Assume $K$ as in the previous proposition and $P(D)$ a constant coefficients differential operator with $\mid \xi \mid^\nu \leq C \mid P(\xi) \mid$, $\quad \mid \xi \mid$ large and some positive constant $C$. Then, $\mid \xi \mid^{\nu+\sigma} \leq C \mid \mathcal{F}_y \, P(D_x)K(x,y) \mid$, $\quad \mid \xi \mid$ large and $\nu > 0$.*

This means, that also $I_{P(D)K}$ is hypoelliptic on $\mathcal{E}'$ (or $B_\alpha$). From the section 11, follows that the iterated polynomial $Q^N_{i_j}$, is reduced with $\sigma \geq 1$, for sufficiently large $N$. This behavior will be stable for further iteration. Thus, the remainder operator, as in (32), is hypoelliptic in $\mathcal{E}'$ (or $B_\alpha$), particularly

$$I_{R_{\lambda,N}}(u) \in C^\infty \Leftrightarrow u \in C^\infty$$

This does however not mean, that it can be represented by a very regular kernel. That is, if $I_K$ is hypoelliptic, then sing supp $(I_K(u)) =$ sing supp $(u)$ and sing supp $(u) =$ sing supp $(I_K(u) - u)$. If $K$ is very regular, $I_K(u) - u = \gamma \in C^\infty$, where we can assume $\gamma$ has nontrivial support. The latter proposition is stronger: Assume for instance $K$ a fundamental solution to a hypoelliptic operator $P$. If $K$ is assumed hypoelliptic, then sing supp $(u - Pu) =$ sing supp $(u)$, for every $u \in \mathcal{D}'$ and if the propositions were equivalent, we would have $P - I$ is regularizing. For $\varphi \in \mathcal{E}'$ (or $B_\alpha$), sing supp $(I_{u_\lambda} - I_{u_{\lambda,N}}(\varphi)) =$ sing supp $(I_{\alpha_\lambda}(\varphi)) \subset$ sing supp $(I_{K^+_\lambda}(\varphi))$ and sing supp $(I_{g_\lambda - K^+_\lambda - u_{\lambda,N}}(\varphi)) =$ sing supp $(I_{K^+_\lambda}(\varphi))$. Thus, to study propagation of singularities, in the formal solution $g_\lambda$, we only have to study a finite number of terms, the remainder will have a hypoelliptic action. If $K$ is the kernel corresponding to a hypoelliptic integral operator $\mathcal{E}' \rightarrow \mathcal{E}'$ and if $U \in \mathcal{E}'(\ \Omega \times \Omega)$, for a bounded open set $\Omega \subset \mathbf{R}^\nu$, then $I_U : C^\infty \rightarrow \mathcal{E}'$. Further, the composition is

defined $I_K(I_U(\varphi))(x) = I_{\big[K,U\big]}(\varphi)(x)$, for $\varphi \in C^\infty$. Let $U = \delta_x$, then using nuclearity for $\mathcal{E}'(\ \Omega)$, we have $\big[K, \delta_x\big] \in \mathcal{E}'(\ \Omega \times \Omega)$ and $x \in \Omega$. Thus, to prove that $R_{\lambda,N}$ is a kernel in $\mathcal{E}'$, it is sufficient to prove that the corresponding integral operator maps $\mathcal{E}' \to \mathcal{E}'$. We will however use a slightly different approach.

12.5. **Some remarks on the generalized Paley-Wiener theorem.** For $K(x,y) \in \mathcal{E}^{'(0)}\ (X \times Y)$ and $X, Y$ open sets in $\mathbf{R}^\nu$, $\mu \in \mathcal{E}'^{(0)}\ (Y)$ we have

$$\int K(x,y)\mu(dy) \in \mathcal{E}'^{(0)}(\mathrm{X})$$

For $K_1, K_2 \in \mathcal{E}'^{(0)}\ (X \times Y)$, further

$$I_{K_1}(I_{K_2}(\mu)) = \int K_1(x,y)\big(\int K_2(y,z)\mu(dz)\big)(dy) = \int\int K_1(x,y)K_2(y,z)dy\mu(dz)$$

this iteration can be repeated infinitely many times, so for $N \geq 0$

$$I_K^N : \mathcal{E}'^{(0)}(\mathrm{Y}) \to \mathcal{E}'^{(0)}(\mathrm{X})$$

Particularly, if $K = F \otimes G$, with $F, G \in \mathcal{E}'^{(0)}$ in $X$ and $Y$ respectively, we have $I_K(\mu)(x) = CF(x)$, with $C = G(\mu)$.

Using (30) and section (7),

$$I_{u_{\lambda,N,(\sim)}}(x,y) = \sum_{|I|=1,I}^{N} [,]'_{|I|+1}(x',y')\varphi \times \ldots \times \varphi(Q_{i_0}\psi, \ldots, Q_{i_{|I|}}\psi)I_{Q_{i_{|I|+1}}\widetilde{\delta_{x''}}}(x'',y'')$$

Now, $\varphi$ can be regarded as a measure in $\mathcal{E}'^{(0)}$ and $|< \varphi, Q_{i_j}\psi >| \leq C \sup_K \mid Q_{i_j}\psi \mid$ for all $j$ and a compact set $K$. We can find a constant $C$, such that $\mid \mathcal{F}''\ I_{u_{\lambda,N,(\sim)}} \mid \leq e^C$ and this estimate still holds if we let $\varphi \to \delta_0$. Let's write $\mathcal{F}''\ I_{u_{\lambda,N,\psi}}$ for this limit. Using the results from section 10, $\mathcal{F}(I_{u_{\lambda,N,\psi}}) = O(1)e^C$, for $\lambda$ large and negative. Using the terminology for the generalized Paley-Wiener theorem, we can say that $\mathcal{F}(\ I_{u_{\lambda,N,\psi}})$ is an entire analytic function of exponential type. Thus $\mid \mathcal{F}\ (I_{u_{\lambda,N,\psi}})(\zeta) \mid \leq e^C$, $\zeta \in \mathbf{C}^\nu$ and that $I_{u_{\lambda,N,\psi}} \in \mathcal{E}'^{(0)}(\mathbf{R}^\nu)$. We note that the same estimate holds, as $N \to \infty$. We assumed above that $x$ was fixed, but a completely analogous argument holds, if we assume instead that $y$ is fixed. We then have that $\mathcal{F}\ (I_{u_{\lambda,N,\varphi}})$ is of exponential type and $I_{u_{\lambda,N,\varphi}} \in \mathcal{E}'^{(0)}(\mathbf{R}^\nu)$.

As $\psi \to \delta_0$ (or analogously if $\varphi \to \delta_0$), we have convergence in $\mathcal{O}_M$ for the Fourier transform and we write $\mathcal{F}(I_{u_{\lambda,N}})$ for the limit. We can prove an estimate for the Fourier-Laplace transform

$$\mid \mathcal{F}(I_{u_{\lambda,N}})(\zeta) \mid \leq C(1 + \mid \zeta \mid)^M e^{H_K(\ \mathrm{Im}\ \zeta)} \qquad \zeta \in \mathbf{C}^\nu$$

for all $N$, where $M \geq 0$ is dependent on $N$, $K$ is a compact, convex subset of $\mathbf{R}^\nu$, $H_K$ the support function for $K$. We say that $\mathcal{F}(\ I_{u_{\lambda,N}}) \in (\ \mathrm{Exp}\ \mathcal{O}_M)$. From Paley-Wiener-Schwartz theorem (([8], part I) , Theorem 7.3.1), it follows that $I_{u_{\lambda,N}} \in \mathcal{E}'(\mathbf{R}^\nu)$. We prove in section 12.6, that $I_{(u_\lambda - u_{\lambda,N})^\sim} \to 0$, as $N \to \infty$. So, using Banach-Steinhaus theorem, it follows that the limit as $N \to \infty$, $I_{u_\lambda} \in \mathcal{E}'(\mathbf{R}^\nu)$. Using appropriate test functions, we have

$$u_\lambda = \sum_{|I|=1}^{N} \sum_{I} \Big[p_{\lambda,i_0,(y'')}, p_{\lambda,i_{|I|+1},(y'')}\Big]'_{|I|+1} \otimes Q_{i_0} \cdots Q_{i_{|I|+1}}\delta_{x''} + R_\lambda \tag{32}$$

Since the convergence is established in the Fréchet space $\mathcal{E}'$, in both variables separately, it follows that we have convergence in $\mathcal{E}'(\mathbf{R}^\nu \times \mathbf{R}^\nu)$.

12.6. **Construction of fundamental solution to $L_\lambda$ in $H'(\mathbf{C}^\nu \times \mathbf{C}^\nu)$.** We have already seen (section 12.3) that, as an iteration operator $\alpha_{\lambda,(\sim)} = O(1)e^{-\kappa|\lambda|^b}$, uniformly on compact sets, as $\lambda \to -\infty$ and this holds for every iteration, that is

$$N^{\alpha,\alpha}(I^N_{\alpha_{\lambda,(\sim)}}) \leq Ce^{-\kappa|\lambda|^b} N^{\alpha,\alpha}(I^{N-1}_{\alpha_{\lambda,(\sim)}}) \leq C' e^{-\kappa(N-1)|\lambda|^b} N^{\alpha,\alpha}(I_{\alpha_{\lambda,(\sim)}}) \leq C'' e^{-\kappa N|\lambda|^b}$$

and we see that $\mid I^N_{\alpha_{\lambda,(\sim)}} \mid \to 0$ as $N \to \infty$, for large negative $\lambda$. The behavior in the "bad" variable, has been estimated to $e^C$, for some positive constant $C$. Thus the remainder $I_{R_{\lambda,M,(\sim)}} = O(1)e^{C-\kappa M|\lambda|^b} \to 0$, uniformly on compact sets, as $M \to \infty$ in $\mathcal{B}_\alpha$, for $\lambda$ sufficiently large and negative. Further $\mid\mid I_{R_{\lambda,M,(\sim)}} \mid - \mid I_{R^{\sim}_{\lambda,M}} \mid\mid \leq C(M,K)$ for some positive constant $C(M,K)$, such that $C(M,K) \to 0$ as $M \to \infty$. That is,

$$\mid\mid I_{R_{\lambda,M,(\sim)}} \mid - \mid I_{R^{\sim}_{\lambda,M}} \mid\mid \leq C_1 \sum_{N \geq M} C^N \mid \big[,\big]'_{N+1} \mid + B_N \mid \big[,\big]'_{N+1} \mid$$

for some positive constants $C_1, C$ and $B_N$. The sum can be estimated with, $e^{-M\kappa|\lambda|^b} \sum_{N \geq M}(C^N + B_N)$ or with $e^{C-M\kappa|\lambda|^b} \to 0$ as $M \to \infty$. Thus $\mid I_{R^{\sim}_{\lambda,M}} \mid \to 0$, as $M \to \infty$. We claim that the fundamental solution $g_\lambda$ is on the form $g_\lambda = K^+_\lambda + \big[u_\lambda, K^+_\lambda\big]$ and to prove that this representation actually holds in $H'$, we prove that the remainder in (33), $R^\sharp_{\lambda,N} \to 0$ in this space.

For a (real) or complex vector space $E$, we can define a Banach space, $Exp_{\alpha,\rho}(E)$, over E with

$$\parallel G \parallel_{\alpha,\rho} = \sup_{z \in E} \mid G(z)e^{-\alpha\rho(z)} \mid$$

where $\alpha \in \mathbf{R}^+$ and $\rho$ is a (real) or complex norm on $E$. We define $Exp(E)$ as the inductive limit of the spaces $Exp_{\alpha,\rho}(E)$. We note the following [13] Ch. 2, Proposition 1.5

*The Fourier-Borel transform establishes a topological vector space isomorphism between the convolution algebra $H'(E)$ and the algebra of entire functions of exponential type on $E^*$, $Exp(E^*)$.*

Assume $R^\sharp_{\lambda,M}$ is represented in $H'$, as a measure $\mu_{M,\epsilon}$, with support in a ball with respect to a complex norm $\rho$ and radius $(1+\epsilon)\alpha$. It follows from the estimates above, that $\sup_z \mid e^{-C}\mathcal{F}\ I_{R_{\lambda,M,(\sim)}} \mid < \infty$, for a positive constant $C$, for every $M$. Here $\mathcal{F}$ denotes the Fourier-Borel transform. The Paley-Wiener-Schwartz theorem, gives $\sup_z \mid e^{-C|z|}\mathcal{F}\ I_{R_{\lambda,M}} \mid < \infty$, for all $M$ and we see that $\mathcal{F}$ $R_{\lambda,M} \in$ Exp $_{C,|z|}$. The constant can be chosen as $C - \kappa M \mid \lambda \mid^b$ and by proposition the cited result in section (12.6), we see that $I_{R_{\lambda,M}} \to 0$ in $H'$ as $M \to \infty$. Obviously, $\mathcal{F}\ R_{\lambda,M} \to 0$ in Exp $_{C,|\xi|}$, that is with a real norm, as $M \to \infty$. We can now assume $\parallel \mu_{M,\epsilon} \parallel \to 0$, as $M \to \infty$. Thus $\mathcal{F}$ $R^\sharp_{\lambda,M}(\zeta) = \int e^{<\zeta,\hat{z}>} d\mu_{M,\epsilon}$ and $\mid \mathcal{F}\ R^\sharp_{\lambda,M}(\zeta) \mid \leq \parallel \mu_{M,\epsilon} \parallel e^{(1+\epsilon)\alpha\rho^*(\zeta)}$, where $\rho^*$ denotes the dual norm, that is $\rho^*(\zeta) = \sup_{|z|\leq 1} |< z, \zeta >|$. We conclude that $\parallel \mathcal{F}$ $R^\sharp_{\lambda,M} \parallel_{(1+\epsilon)\alpha,\rho^*} \to 0$, as $M \to \infty$ and $g_{\lambda,M} \to g_\lambda$ in $H'$.

12.7. **Error in the estimation.** Finally, we need to estimate the difference $u_{\lambda,(\sim)} - \widetilde{u}_\lambda$, where $\widetilde{u}_\lambda$ is $u_\lambda *''_x \varphi *''_y \psi$ and $u_\lambda$ according to (32). Using the representation (32), we see that every term is on the form $\Big[,\Big]' \otimes \text{operator}\delta_{x''}$ so we could put $u_\lambda(x,y) = p_{\lambda,(y'')} \otimes q\delta_{x''}(x,y) + R_\lambda(x,y)$. Thus $\widetilde{u}_\lambda - \widetilde{R}_\lambda = (p_{\lambda,(y'')} \otimes q\delta_{x''}) *''_x \varphi *''_y \psi = (p_{\lambda,(y'')} \otimes q(D_{y''})\delta_{x''}) *''_{x,y} (\varphi\psi)(x,y) =$ $\varphi(x'')\Big(p_{\lambda,(y'')} q(-D_{z''})\psi(y''-z'')\Big)|_{z''=0}$, for $x \in \mathbf{R}^\nu$. The conditions on the

coefficients gives that $\mid u_{\lambda,(\sim),M} - \widetilde{u}_{\lambda,M} \mid \leq C(M,K)$ with $K$ as in the beginning of the section and this relation holds for every $M$. Further, in $H'$,

$$g_\lambda(x,y) = h_\lambda \otimes \delta_{x''}(x,y) + \Big[p_{\lambda,(y'')}, h_\lambda\Big]' \otimes q\delta_{x''}(x,y) + R^\sharp_\lambda(x,y) \tag{33}$$

We have $u_{\lambda,M+1} = \alpha_\lambda + \Big[u_{\lambda,M}, \alpha_\lambda\Big]$. If $g_{\lambda,M}$ is derived from (23) with $u_{\lambda,M}$ instead of $u_\lambda$, then $g_\lambda - g_{\lambda,M} = \Big[u_\lambda - u_{\lambda,M}, K^+_\lambda\Big]$ and if we compare (32) with (33), $R^\sharp_{\lambda,M} = \Big[R_{\lambda,M}, K^+_\lambda\Big]$. For the remainder terms, we get $\mid \Big[R_{\lambda,(\sim),M}, K^+_{\lambda,(\sim)}\Big] - \Big[R_{\lambda,M}, K^+_\lambda\Big]^\sim \mid \leq C_1(M,K)$, with $C_1(M,K) \to 0$ as $M \to \infty$. A calculation similar to the one for $u_{\lambda,(\sim)}$ above, gives $\mid g_{\lambda,(\sim),M} - \widetilde{g}_{\lambda,M} \mid \leq C_2(M,K)$. We conclude that the difference $g_{\lambda,(\sim)} - \widetilde{g}_\lambda = O(1)$ as $\lambda \to -\infty$.

12.8. **Construction of fundamental solution to $P(y, D_{\boldsymbol{y'}}) - \lambda$.** The same method can be used to construct a fundamental solution $K_\lambda(x,y) = k_\lambda \otimes \delta_{x''}(x,y)$ to the variable coefficients operator $P(y, D_{y'}) - \lambda$. The same arguments gives

$$K_\lambda(x,y) = K^+_\lambda(x,y) + \Big[u^0_\lambda, K^+_\lambda\Big](x,y)$$

with $u^0_\lambda$ such that

$$\beta_\lambda = u^0_\lambda - \Big[\beta_\lambda, u^0_\lambda\Big]$$

The Neumann series for $u^0_\lambda$ is particularly simple. Let $\beta_\lambda(x,y) = \sum_j b_{\lambda,j}(x',y') \otimes \delta_{x''}(y'')$ then

$$u^0_{\lambda,t} = \sum_{|I|=1}^{\infty} \sum_I \Big[b_{\lambda,i_0}, b_{\lambda,i_{|I|+1}}\Big]'_{|I|+1} \otimes \delta_{x''} *_x \mathcal{U}_{-2t} *_y \mathcal{U}_{-2t}$$

with convergence in $\mathcal{L}(\mathcal{B}_\alpha)$. If we let $B$ denote the sum, since $(1-\Delta''_x)^t(1-\Delta''_y)^t u^0_{\lambda,t}(x,y) = u^0_\lambda(x,y)$, we get

$$K_\lambda(x,y) = \Big(h_\lambda + \Big[B, h_\lambda\Big]'\Big)(x',y') \otimes \delta_{x''}(y'')$$

## 13. Construction of a parametrix to the operator $L_\lambda$

We have constructed a fundamental solution to the partially formally hypoelliptic operator $L_\lambda$, using a modified "parametrix". We finally prove that use of parametrices corresponding to our operator in the parametrix method would not give a fundamental solution with better properties.

Assume as before, that the singularity is in 0. It is in the parametrix method sufficient to consider contact operators, frozen in 0. As before $L_\lambda = P_\lambda + R$, and let $R^\star(z,D) = \sum \Big[c_j(z) - c_j(z',0)\Big] R_j(D)$ for $R(z,D) = \sum_j c_j(z) R_j(D)$. We now have $L^\star_\lambda = \Big(L^\star_\lambda - P^\Sigma_\lambda\Big) + P^\Sigma_\lambda = A_\lambda + P^\Sigma_\lambda$, where $P^\Sigma_\lambda = P_\lambda(z',0,D')$. Then $A_\lambda(z) = L_\lambda - L^\Sigma_\lambda$, where $L^\Sigma_\lambda$ has constant coefficients on the "bad" side and variable coefficients only for operators strictly weaker or equivalent to $P^\Sigma_\lambda$. Further, $A_\lambda(z',0) = 0$. Assume $K^\Sigma_\lambda$ a fundamental solution to the operator $P^\Sigma_\lambda$. If $L_\lambda$ is assumed partially hypoelliptic, then $-\alpha_\lambda = A_\lambda K^\Sigma_\lambda \in C^\infty(\mathbf{R}^\nu)$, since if $\Sigma = \{z; z''=0\}$ and sing supp $K^\Sigma_\lambda \subseteq \Sigma$, then sing supp $(R - R^\Sigma)K^\Sigma_\lambda \subseteq \Sigma$ but

$R - R^{\Sigma} = 0$ in $\Sigma$, so $(R - R^{\Sigma})K^{\Sigma}_{\lambda} \in C^{\infty}$. Thus $K^{\Sigma}_{\lambda}$ is a parametrix to $L^{\star}_{\lambda}$, that is $L^{\star}_{\lambda}K^{\Sigma}_{\lambda} = \delta_0 - \alpha_{\lambda}$.

For the more general operator, we use the null-space to the remainder operator, ($R^{\star}$ above), in the construction of a parametrix with singular support on $\Sigma$. We can construct a fundamental solution to $L^0_{\lambda} = P^0_{\lambda} + \sum_{j=1}^{r} P_j(0, D')Q_j(D'')$ with singular support (and in fact support) on $\Sigma$, as follows. Let $E_{0,\lambda}$ be the fundamental solution to $P^0_{\lambda}$ and $E_j, j \neq 0$, solutions to the homogeneous equations on $L^2(\mathbf{R}^n)$ (or $\mathcal{E}'$), $P_jE_j = 0$. Let $E_{\lambda}(y) = \sum_{j=0}^{m} E_j(y') \otimes \delta_0(y'')$, then we can prove that $L^0_{\lambda}E_{\lambda} = \delta_0$. We prove the argument for $L^{\Sigma}_{\lambda}$, using ([8] Th. 13.3.3 (1983)). Assuming $P_j(z', D')E_j = 0, j > 0$, $P_{0,\lambda}(z', D')E_{0,\lambda} = \delta_0$, we can find a linear mapping $\mathcal{L}_k : \mathcal{E}' \to \mathcal{E}'$ $(L^2 \to L^2)$, such that for $j > 0$ or $k > 0$, $P_j(z', D')\ \mathcal{L}_k\ P_k(z', D')E_j = P_j(z', D')E_j = 0$, for $E_j$ adjusted to a compact set.

Assume $L^{\Sigma}_{\lambda}$ formally self-adjoint, $L^{\Sigma}_{\lambda} = P^{\Sigma}_{\lambda} + R^{\Sigma}$, where $R^{\Sigma}$ is a tensor product, of a variable coefficients operator and a constant coefficients operator. Let $E_{\lambda} = K^{\Sigma}_{\lambda} + H$, such that $P^{\Sigma}_{\lambda}K^{\Sigma}_{\lambda} = K^{\Sigma}_{\lambda}P^{\Sigma}_{\lambda} = I$ and $R^{\Sigma}H = HR^{\Sigma} = 0$, on $L^2$ $(\mathcal{E}')$. We then have $L^{\Sigma}_{\lambda}E_{\lambda} = I + P^{\Sigma}_{\lambda}H + R^{\Sigma}K^{\Sigma}_{\lambda}$. We now claim that $R^{\Sigma}K^{\Sigma}_{\lambda} = P^{\Sigma}_{\lambda}H = 0$ in $L^2$ $(\mathcal{E}')$. Proof: $R^{\Sigma}$ is surjective on $L^2$ $(\mathcal{E}')$, that is for $f \in L^2(\mathcal{E}')$, there is a $\varphi \in L^2$ $(\mathcal{E}')$, such that $R^{\Sigma}\varphi = f$. Further $L^{\Sigma}_{\lambda}\varphi = P^{\Sigma}_{\lambda}\varphi + f$ and $\varphi = \varphi + HP^{\Sigma}_{\lambda}\varphi + K^{\Sigma}_{\lambda}f$, that is $K^{\Sigma}_{\lambda}f = -HP^{\Sigma}_{\lambda}\varphi$ and $R^{\Sigma}K^{\Sigma}_{\lambda}f = 0$. The second identity is trivial and we have $L^{\Sigma}_{\lambda}E_{\lambda} = I$. Further, $L_{\lambda}E_{\lambda} = L^{\Sigma}_{\lambda}E_{\lambda} + A_{\lambda}E_{\lambda} = I + \beta_{\lambda}$, where $\beta_{\lambda} \in C^{\infty}$ and we have a parametrix to $L_{\lambda}$.

We need to estimate the solutions $E_j$, to the homogeneous equations. Let's consider the bounded mappings $E, E_0, E_1$ on $L^2(\mathbf{R}^n)$ associated to the operator $L_{\lambda}$, according to $P^0_{\lambda}(D')E_0f = f$ for $f \in L^2$, $L^{\Sigma}_{\lambda}Ef = L_{\lambda}(z', 0, D')Ef = f$ for $f \in L^2$ and $(P^0_{\lambda}(D') + R^{\Sigma}(z', D'))E_1f = f$ for $f \in L^2$. Assume $g \in L^2$, such that $R^{\Sigma}(z', D')g = 0$. For $P^{\Sigma}_{\lambda}g = f$, we get $L^{\Sigma}_{\lambda}(z', D')(g - Ef) = 0$ and since $L^{\Sigma}_{\lambda}$ is hypoelliptic, $g = Ef + \eta$ in $L^2$ with $\eta \in C^{\infty}$, such that $L^{\Sigma}_{\lambda}\eta = 0$. Here $R^{\Sigma}$ denotes one of the strictly weaker operators in the representations of $L^{\Sigma}_{\lambda}$ and $g$ corresponds to a solution to the homogeneous equation, for this operator.

Assume $f$ a $L^2-$ function such that $R^{\Sigma}Ef = 0$, is $Ef \in C^{\infty}$? Proof: Immediately $P^{\Sigma}_{\lambda}Ef - f = 0$, extending with $Ef$, we get $(P^{\Sigma}_{\lambda} - I)Ef + (E - I)f = 0$ and conclude that $Ef \in C^{\infty}$. Using that $g = Ef + \eta$, $\eta \in C^{\infty}$, we see that $g \in C^{\infty}$. Further, we can write $P^0_{\lambda}(E_1 - E_0)f = -R^{\Sigma}E_1f$. We have earlier noted that if the left side is assumed in $H^s$ then the right side will be in $H^{s+\sigma}$, for some positive $\sigma$. Repeating this procedure gives $P^0_{\lambda}(E_1 - E_0)f \in C^{\infty}$. We can write $E_1 = E_0B$ where $B = (I + A)^{-1}$ and $A = R^{\Sigma}E_0$. Since $(I + A) : C^{\infty} \to C^{\infty}$, we get $-Af \in C^{\infty}$. Further $Bf \in C^{\infty}$ and finally $f \in C^{\infty}$.

We have earlier proved that

$$\sup \mid D^{\beta'}_{\xi'}\left[\frac{R^0(\xi')}{M(\xi') - \lambda}\right] \mid = O(1) \mid \lambda \mid^{-c} \qquad \lambda \to -\infty \tag{34}$$

for some positive number $c$. We wish to study the relevant commutators over $\mathbf{R}^n$. First, $E_0(y')$ is most easily defined as convolution with a fundamental solution to $P^0_{\lambda}(D')$ (compare [8](1983) Th. 13.2.1). Let $C_{E_0}$ be the commutator $E_0\psi - \psi E_0$, $\psi \in C^{\infty}_0(\mathbf{R}^m)$, such that $\psi = 1$ on some suitable set. A Taylor expansion of order

$m$, for $\psi$ in $x$, gives

$$M_\alpha(C_{E_0} f) \leq C \sup \mid D_{\xi'}^{\beta'} \big[\frac{1}{M(\xi') - \lambda}\big] \mid M_\alpha(f) = O(1) \mid \lambda \mid^{-c} M_\alpha(f)$$

where $\mid \beta' \mid \leq m$ and the norm is taken over $\mathbf{R}^n$. For the commutator $C_A$, the same type of argument gives $N^{\alpha,\alpha}(C_A) = O(1) \sup \mid D_{\xi'}^{\beta'} \big[\frac{R^\Sigma(\xi')}{M(\xi') - \lambda}\big] \mid = O(1) \mid \lambda \mid^{-c}$, as $\lambda \to -\infty$, where $R^\Sigma$ is assumed frozen in some point on $\Sigma$. For the iterated operators, we have $C_{A^n} = A^{n-1} C_A + C_A A^{n-1}$. Using the Banach algebra property of $\mathcal{B}_\alpha$, we can at least say that $N^{\alpha,\alpha}(C_B) = O(1) \mid \lambda \mid^{-c}$ as $\lambda \to -\infty$.

Assume $\psi$ a test function, such that $\psi = 1$ on an open set containing the singularity, then

$$\psi g = E \psi f - C_E f + \psi \eta$$

The commutator $C_{E_1}$, can be rewritten $C_{E_0} B + E_0 C_B$ and is easily estimated in operator norm to $O(1) \mid \lambda \mid^{-c}$. For the mapping $E$, we have the same form $E = F_0 B$, where now $F_0$ is a fundamental solution to the contact operator $L_\lambda^0$, but since this operator is equivalent to $P_\lambda^0$, we get the same estimate for $C_E$ and $E$. All that remains is to estimate $\eta$, a solution to the homogeneous equation corresponding to $P_\lambda^\Sigma$. But this follows immediately from the estimates we have produced for the fundamental solutions to $P_\lambda^\Sigma$. Expressions on the form (34), can be treated as in the proof of Lemma 10.0.1 and we see that

$$N^{\alpha,\alpha}(\ \exp\ (\kappa \mid \lambda \mid^b (z_j' - x_j')) C_A) = O(1) \mid \lambda \mid^{-c} \qquad \lambda \to -\infty$$

with $j, \kappa$ and $\alpha$ as in proposition 5.1. The commutators $C_{E_0}, C_{E_1}, C_E$, can be estimated in the same way. Finally, we must have that $\eta = 0$ in a neighborhood of the singularity, so $\eta = O(1) e^{-\kappa \mid \lambda \mid^b}$, as $\lambda \to -\infty$, uniformly on compact sets in $\mathbf{R}^n$. The fundamental solution $K_\lambda^\Sigma$ was estimated in section 12.8.

**Lemma 13.1.** *If $E$ is parametrix to a differential operator $P$, such that for every $V$( =neighborhood $x$), $PE - \delta_x \equiv 0$ in $V$, then $P$ is not hypoelliptic.*

Proof: Assume $P$ hypoelliptic, with a parametrix $E$. We then have that $(I_E - I)$ is locally regularizing. If locally $I_{PE} = I$, then also locally $u - Pu \in C^\infty$, for all $u \in \mathcal{D}'$. But, since $P$ is hypoelliptic, the same must hold for $P - I$ and we have a contradiction.□

**Remark** : This problem is however easily handled. In the constant coefficients case, we can assume $E$ a fundamental solution to the hypoelliptic operator $P$ and choose a test function $\zeta \in \mathcal{D}$ such that $\text{supp}\ \zeta \cap \text{supp}\ E \neq \emptyset$. Assume further that $\zeta$=1 in $U_\epsilon$ an $\epsilon-$(neighborhood 0), such that the commutator $C_P E \neq 0$ in $\text{supp}\ \zeta \backslash U_\epsilon$, then $\zeta E$ is a parametrix to $P$ and $P \zeta E - \delta_x \neq 0$ in $\mathbf{R}^\nu \backslash U_\epsilon$. In the variable coefficients case, if $P$ is hypoelliptic, then $E$ is very regular. We can assume $\deg P > 0$, so that $E$ has support in some (neighborhood of 0)$\backslash\{0\}$. Any such neighborhood will do. We use that $C_P E \in C^\infty$ and that $I_{PC_E} = -I_{C_P E} \in C^\infty$ and since $P$ is hypoelliptic, $C_E \in C^\infty$.

However, we make a small modification of the parametrix constructed in this section, so that the remainder is regularizing. Assume as before, that $L_\lambda^\Sigma$ is the variable coefficients operator with support on $\Sigma$ and $A_\lambda = L_\lambda - L_\lambda^\Sigma$. Consider, instead of $E_\lambda$, $K_\lambda^\Sigma + K_\lambda^\delta$, where $K_\lambda^\delta = \sum_j E_j \otimes U_j^\delta$, for $U_j^\delta$ very regular distributions, mapping $L^2 \to L^2$ with support on $\Sigma' \times U_\delta$, a neighborhood of $x''$ and with $E_j$, such that $L_\lambda^\Sigma \sum_j E_j = 0$. We use a commutator, to modify the parametrix as before, $C_{K_\lambda^\delta} = K_\lambda^\delta \psi - \psi K_\lambda^\delta$, for some suitable test function $\psi \in C_0^\infty$

and $E^\delta_\lambda = K^\Sigma_\lambda + C_{K^\delta_\lambda}$. We have that $L_\lambda E^\delta_\lambda = L^\Sigma_\lambda K^\Sigma_\lambda + A_\lambda C_{K^\delta_\lambda}$. If in the commutator, the test function is chosen with support on $\Sigma' \times U_\delta$ and $= 1$ on $V_\epsilon \times U_\delta$, for some neighborhood of $x'$,$V_\epsilon \subset \Sigma'$, it follows that $A_\lambda C_{K^\delta_\lambda} \in C^\infty$. This is a parametrix in the usual sense, with regularizing remainder.

## 14. Hypoellipticity in $L^2$ and $\mathcal{D}'$

We have discussed the parametrix construction in $L^2$ and $\mathcal{D}'$. Obviously, an operator homogeneously hypoelliptic in $\mathcal{D}'$, must be homogeneously hypoelliptic in $L^2$. We give the following result in the opposite direction,

**Proposition 14.1.** *Assume $P$ a variable coefficients, constant strength, differential operator, with a representation $P(x,D) = P_0(x,D) + \sum_j R_j(x,D)$, such that $R_j \prec P_0$ and $\sum_j R_j$ with $\sigma > 0$ for all frozen operators. Further, that the operator is defined as a constant coefficients, hypoelliptic type operator outside a compact set and that Re $P \sim P_0$, then $P$ is hypoelliptic in $\mathcal{D}'$.*

Assume however first, that $P = P_0 + \lambda R$, with $R \prec\prec P_0$, a constant coefficients differential operator and that $P_0$ has $\sigma > 0$. It is trivial that, if $\mathcal{N}$ $(P_0 + \lambda I) \neq \{0\}$, for some $\lambda \in \mathbf{C}$ of finite modulus, then $P_0$ is hypoelliptic in $L^2$. According to Fredholm's alternative, the condition is satisfied for all $P_0$. Since $R$ is strictly weaker than $P_0$, also $P$ has $\sigma > 0$. If we can let $P_0 =$ Re P , for the conditions above, the operator $P$ is hypoelliptic in $L^2$. The parametrix construction now gives, for these conditions, a parametrix in $\mathcal{D}'$, with singular support confined to $\Delta$, the diagonal in $\mathbf{R}^\nu \times \mathbf{R}^\nu$.

For a more formal construction, the following lemma is useful.

**Lemma 14.2.** *Given an operator $P_0$, as above with $\sigma > 0$ and $E$ any parametrix, $PE - \delta_x \in C^\infty$ over $\mathcal{D}'$, then $E$ is very regular.*

Proof: We will show that for the commutator $C_E = E\psi_x - \psi_x E$, for a test function in $C^\infty_0(\mathbf{R}^\nu \times \mathbf{R}^\nu)$, $\psi_x = 1$ close to $x$, we have $C_E \in C^\infty$. Assume $PE - \delta_x \in C^\infty$ also over $L^2$. Since $L^2$ with topology induced by $C^\infty$, is a nuclear space, $C_E$ must be defined on $L^2$, by a kernel in $C^\infty$ and this kernel will be regularizing also in $\mathcal{D}'$. We see that, that $Eu - u \in C^\infty$ over $L^2$. Thus, $E\psi_x u - \psi_x u - C_E u \in C^\infty$ over $L^2$ and the result follows for $u \in \mathcal{D}'$.□

**Remark** : Note that it is a consequence of the results in section 11 and proposition 1.3, that a constant coefficients, self-adjoint operator $P$ with $\sigma > 0$, is hypoelliptic in $\mathcal{D}'$, if and only if we have an inequality

$$\| C_P u \|_{H^{0,0}_K} \leq C \| u \|_{H^{0,0}_K(P)} \quad \text{for} \quad u \in H^{0,0}_K(P)$$

where $K$ is such that $C_P \neq 0$ but otherwise arbitrary.
Proof:(of the proposition) Assuming $R = \sum_j R_j$ has $\sigma > 0$ and that the operator is extended with a constant coefficients operator, outside a compact set, means that $R$ is hypoelliptic in $L^2$. The condition that $R \prec P_0$, means that Re $P$ has $\sigma > 0$ and that $P$ is hypoelliptic in $L^2$. Finally, the parametrix construction gives a parametrix in $\mathcal{D}'$, that is very regular, so $P$ is hypoelliptic in $\mathcal{D}'$.□

**Remark** : The importance of the requirement that Re $L \sim L$, is illustrated by the following example (cf.[22]). The operator in $\mathbf{R}^2$ of order $2m + 1$, $m \geq 1$,

$P = (D_x + ix^{2k} D_y)^{2m+1} - ixD_y^{2m}$, is for $k \geq 1$ not hypoelliptic in $\mathcal{D}'$, but it is homogeneously hypoelliptic in $\mathcal{D}'$ and as a consequence, hypoelliptic in $L^2$.

We note, that an operator that does not depend on all variables in space, can be completed to a hypoelliptic operator, $P_{0,t}(D) = P_0(D')(1 - \Delta'')^t$. Let, for a partially hypoelliptic operator, $L = P_0 + R$, $\widetilde{L} = P_{0,t} + R$, where $t$ is chosen such that 2 deg $_{x''} R < t$, that is $R \prec\prec P_{0,t}$. For $\widetilde{L}$, with constant real coefficients, according to what was said above, $\widetilde{L}$ is hypoelliptic in $\mathcal{D}'$. For $\widetilde{L}$, with variable coefficients, such that Re $\widetilde{L} \sim \widetilde{L}$, we also get a hypoelliptic operator in $\mathcal{D}'$. This means that given a variable coefficients, self-adjoint, formally partially hypoelliptic operator, we can always complete this operator to a formally hypoelliptic operator.

**Lemma 14.3.** *The constant, real coefficients operator* $\widetilde{L}(D) = P_{0,t}(D') + R(D)$ *is hypoelliptic in* $\mathcal{D}'$

Proof:
It is sufficient ( and necessary ) to prove, for the commutator $C_{\widetilde{L}}$, defined as $C_{\widetilde{L}} = \widetilde{L}\phi - \phi\widetilde{L} \neq 0$, for a suitable test function $\phi \in C_0^\infty(\mathbf{R}^\nu)$, such that $\phi = 1$ on a small open set in $K$ and $K$ arbitrary, that

$$\parallel C_{\widetilde{L}} u \parallel_{H_K^{0,0}} \leq C \parallel u \parallel_{H_K^{0,0}(\widetilde{L})} \quad \text{for } u \in H_K^{0,0}$$

But since this relation obviously holds for $P_{0,t}$ and since $R \prec\prec P_{0,t}$, implies an even stronger relation, $\parallel Ru \parallel_{H_K^{\sigma',\sigma''}} \leq \parallel u \parallel_{H_K^{0,0}(P_{0,t})}$, for some positive numbers $\sigma', \sigma''$ (Proposition 25.2), the result follows.$\Box$

Note that for a partially hypoelliptic operator $L = P_0 + R$, we have $WF(u) \subset WF(Lu) \cup \Gamma_x$, $u \in \mathcal{D}'$, where $\Gamma_x$ is the characteristic set for the operator $P_0$, that is the set of real zero's to this polynomial. For complex zero's to a constant coefficients, hypoelliptic polynomial, we have with $\xi = \xi' + i\xi''$ and $P(\xi) = 0$, if $\xi''$ is bounded, then $\xi'$ will be bounded. Accordingly, for a constant coefficients, partially hypoelliptic operator, $P(\xi, \eta) = 0$ with $\xi'', \eta$ bounded, implies $\xi'$ bounded (cf.[6], Theorem 1). Let, for frozen $(x, y)$, $\Phi_x(L) = \{(x, y, \xi, \eta); L(x, y, \xi, \eta) = 0 \quad 0 \neq \eta$ bounded $, \xi''$ bounded $\}$. Then, for a partially hypoelliptic operator, the projection of $\Phi_x(L)$ on the real directions, $\xi', \eta'$, is in a bounded set in $\mathbf{R}^n \times \mathbf{R}^n$. For the completed operator, we get $\Phi_x(\widetilde{L}) \subset \subsetneq \neq \Phi_x(L)$, but for the corresponding real sets, $\Gamma_x(\widetilde{L}) = \Gamma_x(L)$. Note that, since the completing polynomial has no real zero's, $\Gamma_x(\widetilde{L})$, is not necessarily bounded.

If the operator is of the form of $L_\lambda^\star$, that is $L_\lambda^\star = L_\lambda^\Sigma + (P_\lambda^\star + R^\star)$, where the two terms have support in complementary sets, then any iterate of the operator is of the same form. That is $(L_\lambda^\star)^2 = (L_\lambda^\Sigma)^2 + B_\lambda$, where $B_\lambda = P_\lambda^\star R^\star + R^\star P_\lambda^\star$ with support outside $\Sigma$ and $(L_\lambda^\Sigma)^N = (L_\lambda^\Sigma)^N + D_\lambda$, and $D_\lambda$ with support outside $\Sigma$. Finally, given an operator $L$, with variable coefficients and constant strength, (but not necessarily self-adjoint), we have that there is an iteration index $N_0$, such that $L^N$ is hypoelliptic for every $N \geq N_0$. Because, Re $L^N \sim$ ( Re $L_{x^0})^N$, where $L_{x^0}$ is the operator $L$ with frozen coefficients and where the right side is hypoelliptic, for $N \geq N_0$, according to Lemma 11.2. According to Lemma 14.2, we thus have that $L^N$ is hypoelliptic for every $N \geq N_0$.

## 15. Propagation of singularities in $\mathcal{D}'$ for the Neumann series

The conditions on $L^{\Sigma}_{\lambda}$, give that $E_{\lambda}$ must be regular in $x'$, that is $\varphi E_{\lambda} \in H^{s,-N}_{K}$, for some $\varphi \in \mathcal{D}$ and $N$ finite. This would mean that the parametrix has the same regularity properties, as a finite development of the fundamental solution constructed in section 12.6. The same method can be used to construct a fundamental solution, using the parametrix we have now constructed. This time $\alpha_{\lambda}$ is replaced by $A_{\lambda}E_{\lambda} \in C^{\infty}$. $E_{\lambda} = K^{\Sigma}_{\lambda} + \sum_j E_j \otimes \delta_{x''}$ with singular support on $\Sigma$. We assume $\Sigma$ adjusted to a singularity in $x$. In section 12.6, we estimated $K^{\Sigma}_{\lambda} = O(1)e^{-\kappa|\lambda|^b} \otimes \delta_{x''}$, as $\lambda \to -\infty$, on compact sets and $K^{\Sigma}_{\lambda} = O(1) \mid \lambda \mid^{-c}$, as $\lambda \to -\infty$, uniformly on $\mathbf{R}^n \times \mathbf{R}^n$. Using the estimates we have produced for the solutions to the homogeneous equations, we have formally $E_{\lambda} = O(1)e^{-\kappa|\lambda|^b} \otimes \delta_{x''}$, on compact sets and $E_{\lambda} = O(1) \mid \lambda \mid^{-c} \otimes \delta_{x''}$, uniformly on $\mathbf{R}^n \times \mathbf{R}^n$. We can write as before $\alpha_{\lambda}(x,z) = \sum_j a_{\lambda,j,(z'')}(x',z') \otimes q_j(x'',z'')$, where $a_{\lambda,j,(z'')} = 0$ on $\Sigma$. Further $v_{\lambda} = \alpha_{\lambda} + \big[\alpha_{\lambda},\alpha_{\lambda}\big] + \ldots = v_{\lambda,N} + R_{v,\lambda,N}$ and $v_{\lambda} \in C^{\infty}_0$ has no support on $\Sigma$.

Assuming as before that the variable $z''$ is frozen in $y'' \neq x''$, then $v_{\lambda}$ will have support on $\Sigma$ and we will get a representation of the fundamental solution, on a form much like the one constructed in section 12.8 and with the same regularity properties. Let's define $\mathcal{D}'_{\Sigma}$-convergence as convergence in $\mathcal{D}'$, such that the singular support, if any, is maintained in $\Sigma$. Then, we obviously have $v_{\lambda,N} \to v_{\lambda}$ in $\mathcal{D}'_{\Sigma}$-meaning. Even for the fundamental solution $g_{\lambda} = E_{\lambda} + \big[v_{\lambda},E_{\lambda}\big]$, we get that $\big[R_{v,\lambda,N},E_{\lambda}\big] \to 0$ in $\mathcal{D}'_{\Sigma}$-meaning, as $N \to \infty$.

**Lemma 15.0.1.** *For a $u \in \mathcal{E}'\ (\mathbf{R}^{\nu})$, $\alpha \in C^{\infty}(\mathbf{R}^{\nu})$ and a constant coefficients operator $P(D)$, there is a $\beta \in C^{\infty}_0(\mathbf{R}^{\nu} \setminus$ sing supp $u)$, such that in $\mathcal{E}'(\mathbf{R}^{\nu} \setminus$ sing supp $u)$*

$$\alpha(P(D)u) = (P(D)\delta_0) * (\beta u)$$

Using that the coefficients to the operator $L$ are in $C^{\infty}$, we can write $L(z,D)u = (L(D)\delta_0) * (\beta u)$, for $u \in \mathcal{E}'$ and $\beta$ an operator corresponding to multiplication with a function in $C^{\infty}(\mathbf{R}^{\nu} \setminus$ sing supp $u)$. That is, assume $E$ the convolution inverse in $\mathcal{E}'$, corresponding to $L(D)\delta_0$. If $L(z,D) = \sum_{j=1}^{r} \alpha_j(z)L_j(D)$, let $L^T(z,D) = \sum_{j=1}^{r} L_j(D)\beta_j(z)$, be an associated operator. We then can find $\beta_j \in C^{\infty}$, such that in $\mathcal{E}'$, $E * L(z,D)u = E * L^T(z,D)u = \sum_{j=1}^{r} \beta_j u = \beta u$. Let's denote the set of multiplication operators, corresponding to a development of an operator $L(z,D)$, $M(L)$. Thus $\beta \in M(L)$, is the set $\{\beta_j\}_{j=1}^{r}$, occurring in the representation of $L^T$. Assume as before $\alpha_{\lambda}(x,z) = A_{\lambda}(z,D)E_{\lambda}(x,z)$, further that $F_{\lambda}$ is the convolution inverse for $L_{\lambda}(D)$, in $L^{\#}_{\lambda} I_{E_{\lambda}} = \delta_x$. Assuming $E_{\lambda}$ a two-sided fundamental solution to $L_{\lambda}(y,D_y)$, this means that $L^{\#} = {}^tL_{\lambda}$. According to Lemma 15.0.1, we can find a $\gamma \in M({}^tL_{\lambda})$, with support outside $\Sigma$, such that $\gamma I_{E_{\lambda}}(u) = F_{\lambda} * u$, $u \in C^{\infty}_0(\mathbf{R}^{\nu})$. Using the Lemma 15.0.1 one more time, gives $\gamma I_{\alpha_{\lambda}}(u) = F_{\lambda} * A_{\lambda}(D)\delta_0 * \beta u$, $u \in \mathcal{E}'$, now assuming $\gamma$ with support outside the singular support for $I_{E_{\lambda}}(u)$ and $\beta \in M({}^tA_{\lambda})$ with support outside sing supp $u$. Note that the operator parts in the operators $A_{\lambda}$ and ${}^tL_{\lambda}$ are the same, so outside $\Sigma$, the operator $I_{\alpha_{\lambda}}$ acting on $C^{\infty}_0$, corresponds to multiplication with test functions or we could say that it has only the localizing property.

Assume $E_{\lambda,\delta} = K^{\Sigma}_{\lambda} + \sum_j E_j \otimes T_{\delta}$, where $T_{\delta}$ is a very regular measure in $\mathcal{E}'^{(0)}$ with support contained in an open set $U_{\delta}$, such that $U_{\delta} \to \{x''\}$, as $\delta \to 0$. We then

have, for a fixed $x$, sing supp $E_{\lambda,\delta}(x,\cdot) \subset \Sigma \cup \{x'\} \times U_\delta$. Since the coefficients in $\alpha_{\lambda,\delta}$ are 0 on $\Sigma$, we have sing supp $\alpha_{\lambda,\delta}(x,\cdot) \subset \{x'\} \times U_\delta$. Let's write $Z_\delta = \{x'\} \times U_\delta$. The support for $\alpha_{\lambda,\delta}(x,\cdot) \subset W' \times U_\delta$, where $W' = \{y'; y \in W\}$ and where $W$ is a compact set, in which the operator is formally partially hypoelliptic. Let's assume, that for the situation where $y''$ is fixed outside $\Sigma$, $\alpha_{\lambda,(y''),\delta}$ is a measure and study how iteration of $I_{\alpha_\lambda,(y''),\delta}$, corresponds to convolution. Using the Lemma 15.0.1, on the tensorized integral operators, and writing formally $A_\lambda(z', y'', D) = P_\lambda(z', D') \otimes Q_\lambda(D'')$, we have $\gamma_1 I'_{\alpha_{\lambda,\delta}}(v_1) = F_\lambda * P_\lambda(D')\delta_0 * \beta_1 v_1$, $v_1 \in C_0^\infty(\mathbf{R}^n)$ and $\gamma_2 I''_{\alpha_{\lambda,\delta}}(v_2) = G * Q(D'')\delta_0 * \beta_2 v_2$, $v_2 \in C_0^\infty(\mathbf{R}^m)$ or equivalently $(F_\lambda \otimes G) * A_\lambda(D)\delta_0 *' \beta_1 v_1 *'' \beta_2 v_2$. We assume that $\gamma_1 \in M({}^tP_\lambda)$, $\gamma_2 \in M({}^tQ)$, $\beta_1 \in M({}^tP_\lambda)$ and $\beta_2 \in M({}^tQ)$. We can again identify the operator parts, and we have $\gamma_1 \otimes \gamma_2 I_{\alpha_{\lambda,\delta}}(v_1 \otimes v_2) = \delta_0 *' \beta_1 v_1 *'' \beta_2 v_2$. Iteration of this procedure, gives for test-functions $\varphi \otimes \psi \in C_0^\infty(\mathbf{R}^\nu)$,

$$\gamma^{(N)} I^N_{\alpha_{\lambda,\delta}}(\varphi \otimes \psi) = \delta_0 * \gamma^{(N-1)} I^{N-1}_{\alpha_{\lambda,\delta}}(\varphi \otimes \psi) = \ldots = \gamma^{(0)} \varphi \otimes \psi$$

for $\gamma^{(j)} \in M$ on tensor form, $j = 0, \ldots, N$ and the iteration can be repeated infinitely many times.

## 16. Propagation of singularities in measure topology

Assume $\mu \in \mathcal{E}'^{(0)}(\mathbf{R}^\nu)$, is such that $\widehat{\mu}$ slowly decreasing. Using the conditions on $\alpha_{\lambda,\delta}$ and the Paley-Wiener theorem, we have that $\widehat{I_{\alpha_{\lambda,\delta}}(\mu)}/\widehat{\mu}$ is an entire analytic function. Using a theorem in [7],(Theorem 3.6), we have a $F_\Sigma \in \mathcal{E}'$, such that $F_\Sigma * \mu = I_{\alpha_{\lambda,\delta}}(\mu)$. We can assume $F_\Sigma$ on tensor form. Further $I^N_{\alpha_{\lambda,\delta}}(\mu) = F_\Sigma * \ldots * F_\Sigma * \mu$, where the convolution is repeated $N$ times. The singular support for this representation, is contained in $\{x_1 + x_2 + \ldots + x_N + y; x_j \in \text{ sing supp } F_\Sigma, y \in \text{ sing supp } \mu\}$. For a fixed $x$ away from 0, the singular support for $F_\Sigma$ is included in $W' \times U_\delta$. In $\mathcal{E}'^{(0)}$, we have the equalities for the measures under hand

$$F_\Sigma * (\mu_1 \otimes \mu_2)(x) = F_\Sigma * (\mu_1 \otimes \delta_0) * (\delta_0 \otimes \mu_2) = F_\Sigma *' \mu_1 *'' \mu_2$$

although their singular supports may differ. We can use the tensor form of $F_\Sigma$ and these equalities, to get $I^N_{\alpha_{\lambda,\delta}}$ as an iteration of partial convolutions. This gives a particularly simple displacement of the singular support for $\mu = \mu_1 \otimes \mu_2$. Note that $I_{L^\Sigma_\lambda(y,D)E_\lambda}(\mu) = \mu$, so assuming ${}^tL_\lambda = L_\lambda$ and using that the differential operator part in $L^\Sigma_\lambda$ is $L_\lambda$, this means that $I_{c_\Sigma E_\lambda}(\mu) = F_\lambda * \mu$, where $c_\Sigma$ denotes multiplication with $C^\infty-$ functions, derived from the coefficients in $L^\Sigma_\lambda$ as before. Further, if $cE_\lambda$ denotes multiplication with the $C^\infty$-functions corresponding to the coefficients in $L_\lambda(y, D)$, we have $I_{cE_\lambda}(\mu) = F_\lambda * \mu - F_\lambda * I_K(\mu)$, where $K$ is regularizing and we shall see that the last term does not effect the wave front set, that is $WF(I_{cE_\lambda}(\mu)) = WF(F_\lambda * \mu)$.

**Lemma 16.0.2.** *Assume $X, Y$ open sets in $\mathbf{R}^\nu$, and $X' \times X'' \subset X$, $Y' \times Y'' \subset Y$. Given a measure $\mu \in \mathcal{E}'^{(0)}$ $(X \times Y)$, (but not the Dirac-measure), with $\widehat{\mu}$ slowly decreasing, such that $\mu = \mu_1 \otimes \mu_2$, for $\mu_1, \mu_2$ very regular in $X' \times Y'$ and $X'' \times Y''$ respectively, iteration of partial convolution with the convolution kernel corresponding to $\mu_1$, followed by the kernel corresponding to $\mu_2$, will for $x$ fixed sufficiently far away from 0, outside the diagonal after a finite number of steps, give a $C^\infty$-function.*

**Remark** : The conditions on $\mu$ are sufficient to conclude that $\mu$ is a parametrix to a partially hypoelliptic differential operator. Assume $\mu \in \mathcal{E}'^{(0)}$ invertible and otherwise according to the conditions. Then we have existence of the formal

inverse in $\mathcal{E}'^{(0)}$, why we can solve the equation $\big[\mu - 1\big] * f = w \in C^\infty$, according to $\sum \mu^j * w = \sum_0^N + \sum_{N+1}^\infty$, where the last term is in $C^\infty$ and by localization we can assume $f \in \mathcal{E}'$. Now choose the polynomial $P$ as a partially hypoelliptic polynomial with $\Delta_{\mathbf{C}}(P) \subset Z_{\widehat{f}}$ (lineality). We then have

$$\text{sing supp } (\mu * f) = \text{ sing supp } (f)$$

$$\text{sing supp } (P(D)f) = \text{ sing supp } (f)$$

Chose $g \in \mathcal{D}'_{L^1}$ such that $g$ is hypoelliptic in $\mathcal{D}'_{L^\infty}$ and such that $P(D)g * f - \delta \in C^\infty$ ($\Rightarrow g * f - \delta \in C^\infty$), then

$$P(D)\mu - \delta \sim P(D)g * \mu * f - \delta \sim$$

$$\sim P(D)g * \big[\mu * f - f\big] + g * \big[P(D)f - f\big] + \big[g * f - \delta\big] \in C^\infty$$

where $\sim$ indicates that the singular supports coincide. Note that if we assume $P^N g * f - \delta \in C^\infty$ we also have $g * \mu^N * \sum_0^N \in C^\infty$ and the proposition is that there exists a $g$, hypoelliptic for convolution, such that
ch sing supp $(\mu^N * \sum_0^N) \subset -$ch sing supp $g$

Proof: Let's assume $\mu = E_{\lambda,\delta}$, an invertible measure corresponding to $\alpha_{\lambda,\delta}$, according to section 15. Using the translation invariance for constant coefficients operators, we first assume that $L_\lambda \delta_0 * F_\lambda = \delta_0$. By tensorizing the operator $L_\lambda$, we have that $F_\lambda$ has support only in $\{y_1 \geq 0, y_2 \geq 0, \ldots, y_\nu \geq 0\}$. The displacement of the support and singular support, during iterated convolution, will be only in the direction of non-negative coordinates. Assume $Z$ an open set containing the singularities for $F_\lambda$. The sets $Z$,$Z+Z$,$Z+Z+Z$,..., then constitute a countable covering of the support for the iterated convolution. Since this support is compact, we must have a finite sub covering.

The singular support for $F_\lambda$ is included in $\cup\{y; y_j = 0$ for some $j\}$. Using that $E_{\lambda,\delta}$ is invertible, this means that ch sing supp $F_\lambda^N * E_{\lambda,\delta} =$
ch sing supp $F_\lambda^{N+1} * E_{\lambda,\delta}$, for some $N$, where the left side is
ch sing supp $F_\lambda * \ldots * F_\lambda * E_{\lambda,\delta}$, with $N$ repeated $F_\lambda's$.

As we translate the singularity to some $x$ far away from 0, we would prefer to write

$$L_\lambda(D)\delta_0 * F_\lambda(x - y) = \delta_0(x - y)$$

Replace $x$ with a point, close to the diagonal, but $\neq x$. This corresponds to a displacement of the support for $F_\lambda$, away from the coordinate axes. Denote the corresponding kernel $E^x_{\lambda,\delta}$. The singular support for $E^x_{\lambda,\delta}(z, \cdot)$, will during the iterated convolution with $F_\lambda$, be moved along the diagonal in the direction of negative coordinates. We still have ch sing supp $F^N_{\lambda,x} * E^x_{\lambda,\delta} =$ ch sing supp $E^x_{\lambda,\delta}$, which after a finite number of iterations is a contradiction. That is, outside the diagonal, we have $F^N_{\lambda,x} * E_{\lambda,\delta} \in C^\infty$. $\square$

**Remark** : If $y \in$ ( neighborhood $x$), for instance $\mid x - y \mid < \epsilon$, for some $\epsilon > 0$. Assume $z$ a point on the distance $\epsilon$ from the origin and with only positive coordinates. Then $y - 2\epsilon z \notin$ ( neighborhood $x$). This is the type of translation suggested in the proof above.

## 17. Asymptotic convergence for the Neumann series $v_\lambda$

Assume $N$, the smallest positive integer, such that the singular support is stable, that is ch sing supp $v_{\lambda,N,\delta} =$ ch sing supp $v_{\lambda,N+1,\delta}$, then Lemma 16.0.2 applied to the operator $A_{\lambda,\delta}$ gives that, since the convolution operator corresponding to $A_{\lambda,\delta}$, has no support on $\Sigma$ and since this set contains the diagonal, we have $v_{\lambda,N} \in C^\infty(\mathbf{R}^\nu \times \mathbf{R}^\nu)$. Thus,

$$L_\lambda(E_{\lambda,\delta} + \big[v_{\lambda,N+1,\delta}, E_{\lambda,\delta}\big]) = \delta_x + \alpha_{\lambda,\delta} + v_{\lambda,N+1,\delta} + \big[v_{\lambda,N+1,\delta}, \alpha_{\lambda,\delta}\big] + \text{ a term in } C^\infty$$

Using that $v_{\lambda,N+2,\delta} = \alpha_{\lambda,\delta} + \big[v_{\lambda,N+1,\delta}, \alpha_{\lambda,\delta}\big]$, we see that we have in fact a parametrix to the operator $L_\lambda$.

Let's write $v^{\otimes}_{\lambda,k}$, for the tensorized iteration. We then have

$$WF(L_\lambda(E_{\lambda,\delta} + \big[v^{\otimes}_{\lambda,k,\delta}, E_{\lambda,\delta}\big]) - \delta_x) \subset WF(L_{\lambda,(y'')}(E_{\lambda,\delta} + \big[v^{\otimes}_{\lambda,k,\delta}, E_{\lambda,\delta}\big]) - \delta_x)$$

and this would work just as well as a parametrix.

Let's write $v^{p}_{\lambda,k}$, for the iterated partial convolution described above. We then have

$$WF(\alpha_{\lambda,\delta}) \subset WF(\alpha^{\otimes}_{\lambda,\delta}) \subset WF(\alpha^{p}_{\lambda,\delta})$$

On $\Sigma$, we have $E_{\lambda,\delta} + \big[v_{\lambda,\delta}, E_{\lambda,\delta}\big] \to E_\lambda$, as $\delta \to 0$ with convergence in $H'$. Using the representation $g_{\lambda,\delta} = E_{\lambda,\delta} + \big[v^{p}_{\lambda,N,\delta}, E_{\lambda,\delta}\big] + R^{p}_{\lambda,N,\delta}$, with $R^{p}_{\lambda,N,\delta} \in C^\infty$ in a sufficiently small neighborhood of the diagonal, we have convergence in $C^\infty$, for all but a finite number of terms, as $\delta \to 0$.

Finally, we have earlier established convergence in $\mathcal{E}'$, for the equivalent to $v^{\otimes}_{\lambda}$. In this case we have that in the "bad" variable, any direction may be singular for $v^{\otimes}_{\lambda,N}$, while the in "good" variable, we have a hypoelliptic situation. We say, for $v^{\otimes}_{\lambda,N} \in \mathcal{D}'_\Gamma$, that $v^{\otimes}_{\lambda,N} \to v^{\otimes}_{\lambda}$ in $\mathcal{D}'_\Gamma$-meaning, if the convergence holds in $\mathcal{D}'$, while the wave front set is contained in $\Gamma$. Thus we must have

$$\sup_V \mid \xi \mid^N \mid (\varphi v^{\otimes}_{\lambda,N} - \varphi v^{\otimes}_{\lambda}\widehat{)} \mid \to 0$$

for $N = 1, 2, \ldots$ and $\varphi \in C^\infty_0($ neighborhood $W)$, such that $\Gamma \cap ($ supp $\varphi \times V) = \emptyset$ for any closed cone $V$. But since $V$ only can contain "good" directions, the convergence follows immediately, from what has already been proved. The same result can be given for the fundamental solution.

## 18. Asymptotic hypoellipticity for the operator $L_\lambda$

We can now give a definition of asymptotic hypoellipticity. Assume $g_\lambda$ a fundamental solution to the partially formally hypoelliptic, formally self adjoint, variable coefficients operator $L_\lambda$. Assume further that we have $g_\lambda = E_{\lambda,\delta} + \Big[v_{\lambda,N,\delta}, E_{\lambda,\delta}\Big] + R_{\lambda,N,\delta} = g_{\lambda,N} + r_{\lambda,N}$. According to Lemma 16.0.2, we have for $N$ sufficiently large, $r_{\lambda,N} \in C^\infty$. We first give the following lemma.

**Lemma 18.0.3.** *If $L_\lambda$ is hypoelliptic, then $g_{\lambda,N}$ is very regular, for every $N \geq 0$. Conversely, if $L_\lambda$ is partially formally hypoelliptic and $g_{\lambda,N}$ is hypoelliptic with $r_{\lambda,N} \in C^\infty$, for every $N \geq 0$, then $L_\lambda$ is hypoelliptic.*

Proof:
Assume $L_\lambda$ hypoelliptic. We have $L_\lambda(g_\lambda - r_{\lambda,N}) = \delta_x - L_\lambda r_{\lambda,N}$, for every $N \geq 0$. It is a trivial consequence of Lemma 16.0.2, that $L_\lambda r_{\lambda,N} \in C^\infty$, for every $N \geq 0$. Thus $g_\lambda - r_{\lambda,N}$ is a $\mathcal{D}'-$ parametrix to the operator $L_\lambda$, which must be very

regular. Assume now $g_\lambda - r_{\lambda,N}$ hypoelliptic with $r_{\lambda,N} \in C^\infty$, then $g_{\lambda,N}$ is a parametrix to $L_\lambda$ and

$$\text{sing supp } I_{(g_\lambda - r_{\lambda,N})} L_\lambda \mu = \text{ sing supp } \mu \quad \text{ for every } \mu \in \mathcal{D}'$$

and since $g_{\lambda,N}$ is hypoelliptic, sing supp $L_\lambda \mu =$ sing supp $\mu$. $\square$

**Definition 18.0.4** ( Asymptotically hypoelliptic operator )**.** *Assume $g_\lambda = g_{\lambda,N} + r_{\lambda,N}$ a fundamental solution to a partially formally hypoelliptic operator as in the previous lemma. If, for a finite positive integer $N_0$, we have that $r_{\lambda,N} \in C^\infty$ and $g_{\lambda,N}$ hypoelliptic, for all $N \geq N_0$, then the operator $L_\lambda$ is said to be asymptotically hypoelliptic.*

We could also say that, for an asymptotically hypoelliptic operator, we have, for large $N$, $L_\lambda r_{\lambda,N} \in C^\infty$. Thus $L_\lambda(g_\lambda - g_{\lambda,N}) \to 0$ as $N \to \infty$ in $C^\infty$ and using an inequality for constant strength operators, $g_\lambda - g_{\lambda,N} \to 0$ in $C^\infty$, as $N \to \infty$. So, $g_\lambda$ is approximated by parametrices, asymptotically in $C^\infty$.

**Proposition 18.0.5.** *Given a partially formally hypoelliptic operator $L_\lambda(y, D_y) = P_\lambda(y, D_{y'}) + \sum_{j=1}^r P_j(y, D_{y'}) Q_j(D_{y''})$, we have a parametrix, $E_\lambda$, on the form $K_\lambda^\Sigma + \sum_j E_j \otimes \delta_{x''}$ with singular support on $\Sigma = \{z \in \mathbf{R}^\nu;\ z'' = x''\}$. Here $K_\lambda^\Sigma$ is the fundamental solution to the operator $P_\lambda(y', x'', D_{y'})$ and $E_j$ solutions to the homogeneous equations $P_j(y', x'', D_{y'}) E_j = 0$ for $j = 1, \dots, r$. We have the estimates $E_\lambda = O(1) e^{-\kappa |\lambda|^b} \otimes \delta_{x''}$, as $\lambda \to -\infty$, on compact sets in $\mathbf{R}^\nu \times \mathbf{R}^\nu$. Further $E_\lambda = O(1) \mid \lambda \mid^{-c} \otimes \delta_{x''}$ uniformly on $\mathbf{R}^n \times \mathbf{R}^n$, as $\lambda \to -\infty$.*

## 19. Conclusions concerning $g_\lambda$

The fundamental solution to the constant coefficients operator, $K_\lambda^+$, is obviously of exponential $\rho^*$-type 0, in the bad variable, over $\mathbf{C}^\nu$. For $K_\lambda^+ = T_1 \otimes T_2$, we have

$$\mid \widehat{T}_1(i\xi') \mid \leq C \sup \mid [\frac{1}{M(-\xi') - \lambda}] \mid$$

Thus $K_\lambda^+$ is of exponential $\rho^*$-type 0, also in the "good" variable. According to [13] Ch. 2, Corollarium 2, this means that it allows real support. For $u_\lambda$, we established in section 12.5 that it is in $\mathcal{E}'$, which means that it can be represented as an analytic functional, by a measure with compact support in $E = \mathbf{C}^\nu$. We will use the same notation $u_\lambda$ for these elements. Let's assume that it is portable by a ball with respect to a complex norm $\rho$ and of radius $\alpha$. According to [13] Ch. 2, Lemma 1, this means that it is of exponential $\rho^*$-type $\leq \alpha$. The problem is to establish whether also $u_\lambda$ allows real support, in which case the same holds for the representation of $g_\lambda$ as analytic functional. Immediately, if $E^*$ is instead a compact set in $\mathbf{C}^\nu$, we have that $u_\lambda$ is of exponential $\rho^*$-type 0 and allows real support. Also, it is of exponential $\rho^*$-type $\alpha$ and allows real support in $H'(\mathbf{R}^\nu)$.

In the general case, we can at least say, using [13] Ch.2, Proposition 1.2, if $E_{\mathbf{R}}$ a real vector space with complexification $E_{\mathbf{C}}$, $\mathcal{D}$ $(E_{\mathbf{R}})$ is a dense sub algebra of $H'(E_{\mathbf{C}})$ and

$$u_\lambda(\varphi) = \int_{\mathbf{R}^\nu} \psi(x) \varphi(x) dx \qquad \varphi \in H(E_{\mathbf{C}}), \psi \in \mathcal{D}(E_{\mathbf{R}})$$

The conclusion so far, is that $g_\lambda$ exists in $H'(E)$, $E = \mathbf{C}^\nu$ and is portable by a ball of radius $\alpha$, with respect to a complex norm $\rho$. Further, as $B_{\rho,\alpha}$ can be regarded as an analytic variety, using [13] (Ch.I Cor. to Th. 2.5), there exists a unique

$H(E)$-convex set $W$ in $B_{\rho,\alpha}$, containing the $H(E)-$convex hull to the support for $g_\lambda$.

For the next proposition we need:

$$G_{x,\lambda}^{(\alpha',\alpha')}(x',y') = \frac{1}{(2\pi)^n}\int \frac{\xi^{2\alpha'}\exp(i(x'-y')\cdot\xi')}{\mathrm{Re}P^x(\xi')-\lambda}d\xi'$$

where we assume the operator $P^x$ formally self-adjoint in $L^2(\mathbf{R}^\nu)$. We write $g_\lambda^{(\alpha,\beta)}(x,y)$ for the derivative $(iD_x)^\alpha(iD_y)^\beta g_\lambda(x,y)$. We note that the coefficients corresponding to $L_\lambda$, can be extended to entire analytic functions, using $\mid \widehat{c}_\alpha(z) \mid\leq Ce^{c\mid \text{ Im } z\mid}$ and the condition on self-adjointness means that the symbol $L_\lambda(z,\zeta)$ can be treated as real analytical. Using [17] Lemma 10 we have

**Proposition 19.0.1.** *For any positive integer $M$, provided $\varrho$ (as in section 3.2) $> n+M$, if $\varphi,\psi\in C_0^\infty(\mathbf{R}^m)$ with support in a neighborhood of the origin, then there is for all sufficiently large negative values of $\lambda$, an analytic functional $g_\lambda$ on $\mathbf{C}^\nu\times\mathbf{C}^\nu$ and on $\mathbf{R}^\nu\times\mathbb{R}^\nu$, with the following properties:*

1. *for every $x\in\mathbf{R}^\nu$ the analytic functional on $\mathbf{R}^\nu$, $g_\lambda(x,\cdot)$ is a fundamental solution with singularity $x$ to the operator $L(y,D_y)-\lambda$*
2. *$g_\lambda$ is in $H'$ on the form $K_\lambda^+(x,y)+\big[u_\lambda,K_\lambda^+\big](x,y)$, where $u_\lambda(x,y)$ can be represented in $\mathcal{E}'(\mathbf{R}^\nu\times\mathbf{R}^\nu)$, as an infinite sum of tensor products*

$$u_\lambda=\sum_{|I|=1}^\infty\sum_I\Big[\Big[p_{\lambda,i_0,(y'')},p_{\lambda,i_{|I|+1},(y'')}\Big]'_{|I|+1},h_\lambda\Big]'\otimes Q_{i_0}\dots Q_{i_{|I|+1}}\delta_{x''}$$

3. *$g_\lambda *_x'' \varphi *_y'' \psi$ belongs to $C^M(\mathbf{R}^\nu\times\mathbf{R}^\nu)$ and for every multi-order $\alpha$ with $2\mid\alpha'\mid\leq M$, we have with some positive constant $c$*

$$g_\lambda^{(\alpha,\alpha)} *_x'' \varphi *_y'' \psi(x,x) = (1+O(1)\mid\lambda\mid^{-c})G_{x,\lambda}^{(\alpha',\alpha')}\Big[(iD_{x''})^{\alpha''}\varphi\Big]\Big[(iD_{y''})^{\alpha''}\psi\Big](x,x)$$

$\lambda\to-\infty$, *for every* $x\in\mathbf{R}^\nu$.

If we, in the argument following (33) use the estimates for $K_\lambda^+$, that are proven in [17] Lemma 10 (compare with the third item in the following proposition), we get in the first two, (still according to [17] Lemma 10)

**Proposition 19.0.2.** *With conditions as in Proposition 19.0.1,*

1. *$g_\lambda *_x'' \varphi *_y'' \psi(x,y)=O(1)\mid\lambda\mid^{-c}$, $\quad\lambda\to-\infty$ uniformly on $\mathbf{R}^\nu\times\mathbf{R}^\nu$, for some positive constant $c$.*
2. *for $\mid\alpha'\mid\leq M$, then $D_x^\alpha g_\lambda *_x''\varphi *_y''\psi(x,\cdot)\in C^\infty(\{y\in\mathbf{R}^\nu;\ y-x\notin 0\times \text{supp }\psi\})$, for every $x\in\mathbf{R}^\nu$. Further, for all multi-index $\alpha,\beta$ , $g_\lambda^{(\alpha,\beta)} *_x''\varphi *_y''\psi(x,y)=O(1)\exp(-\kappa\mid\lambda\mid^b)$, $\lambda\to-\infty$, uniformly on compact subsets in $\mathbf{R}^\nu\times\mathbf{R}^\nu$, where $\kappa$ is a positive constant that may depend on the compact subset, and where $b$ is the positive number corresponding to $M$ as in (15)*
3. *For the fundamental solution corresponding to the operator $P(y,D_{y'})-\lambda$, we have the estimates, $K_\lambda(x,y)=O(1)\mid\lambda\mid^{-c}\otimes\delta_{x''}$ as $\lambda\to-\infty$, uniformly on $\mathbf{R}^\nu\times\mathbf{R}^\nu$, for some positive constant $c$. Further for $\mid\alpha'\mid\leq M$, $D_x^{\alpha'}\Big(h_\lambda+\Big[B,h_\lambda\Big]\Big)(x',\cdot)\in C^\infty(\mathbf{R}^n\setminus x')$. For all multi-index $\alpha',\beta'$, $K_\lambda^{(\alpha',\beta')}(x,y)=O(1)\exp(-\kappa\mid\lambda\mid^b)\otimes\delta_{x''}$, as $\lambda\to-\infty$, uniformly on compact sets in $\mathbf{R}^\nu\times\mathbf{R}^\nu$ with $\kappa$ and $b$ as in 10.2.2. Finally, $K_\lambda^{(\alpha',\alpha')}(x,x)=(1+O(1)\mid\lambda\mid^{-c})G_{x,\lambda}^{(\alpha',\alpha')}\otimes\delta_{x''}(x,x)$ as $\lambda\to-\infty$, for every $x\in\mathbf{R}^\nu$*

## 20. Homogeneously hypoelliptic operators

We will as usual use the same notation for the Schwartz kernel and its corresponding integral operator. This means for $L_\lambda F_\lambda = \delta_x - \gamma$ and $\gamma \in C^\infty$, that $L_\lambda F_\lambda \in \Phi$, that is it is Fredholm. Using standard arguments from the Fredholm theory, we can assume that both $L_\lambda$ and $F_\lambda$ are Fredholm operators.

We consider the standard projections $P : L^2 \to R(L_\lambda)$, $Q : H^{s,t} \to N(L_\lambda)$. $L_\lambda \in \Phi(H^{s,t}, L^2)$ gives a decomposition $H^{s,t} = X_0 \bigoplus N(L_\lambda)$ and $L^2 = Y_0 \bigoplus R(L_\lambda)$, where $N(L_\lambda)$ denotes the solutions to the homogeneous equation and $R(L_\lambda)$ the range of $L_\lambda$. We can construct an inverse, $E_\lambda$, to $L_\lambda$, considered as an operator on $X_0$, which is extended on $Y_0$ to an operator in $B(L^2, H^{s,t})$. Using a fundamental theorem in the Fredholm theory, there is a $E_\lambda \in \Phi(L^2, H^{s,t})$, such that $E_\lambda L_\lambda = I$ on $X_0$ and $L_\lambda E_\lambda = I$ on $R(L_\lambda)$. Let $P^\perp = (I - P)$, then $P^\perp(L^2) = N(E_\lambda)$. Further, $L_\lambda E_\lambda(I - P^\perp) = (I - P^\perp)$ or $L_\lambda E_\lambda = I - P^\perp$, and $P^\perp$ is a finite rank operator (a compact operator). In the same way $E_\lambda L_\lambda(I - Q) = (I - Q)$, or $E_\lambda L_\lambda = I - Q$. We note that, for $\sigma > 0$, $L_\lambda$ is homogeneously $L^2$-hypoelliptic, which means that $N(L_\lambda) = Q(H^{s,t}) \subset C^\infty$, that is $Q$ is regularizing on $H^{s,t}$ and $E_\lambda$ is a left parametrix to $L_\lambda$. If $L_\lambda$ is homogeneously $L^2$-hypoelliptic, then also its Hilbert-space adjoint is homogeneously $L^2$-hypoelliptic, that is $P^\perp(L^2) = N(L_\lambda^{\text{adj}}) \subset C^\infty$, and $P^\perp$ is regularizing on $L^2$. We conclude that $E_\lambda$ is a left and right parametrix to the operator $L_\lambda$.

Noting that $L_\lambda E_\lambda = I - P^\perp$ in $L^2$ with $P^\perp$ regularizing, (we are assuming the projections non-trivial) we have that $\text{sing supp}_{L^2}(L_\lambda E_\lambda \varphi) = \text{sing supp}_{L^2}(\varphi)$. Further, $P = I - P^\perp$, means that $P$ is hypoelliptic on $L^2$. The same observations, can be made for $E_\lambda L_\lambda = I - Q$, so $Q^\perp$ is hypoelliptic. Finally, $\text{sing supp}_{L^2}(\varphi) = \text{sing supp}_{L^2}(L_\lambda E_\lambda \varphi) \subset \text{sing supp}_{L^2}(E_\lambda \varphi)$, so $E_\lambda$ is hypoelliptic. Also, $\text{sing supp}_{L^2}(\varphi) = \text{sing supp}_{L^2}(E_\lambda L_\lambda \varphi) \subset \text{sing supp}_{L^2}(L_\lambda \varphi)$ and we conclude,

**Proposition 20.0.1.** *On $L^2$,any homogeneously hypoelliptic operator $L_\lambda$ is hypoelliptic and conversely.*

The extension of $E_\lambda$ to $R(L)$ can be made in different ways. If $L_\lambda$ is assumed homogeneously $L^2$-hypoelliptic, then $E_\lambda$ can be defined as regularizing on $R(L)$. Considered as an operator on $L^2$, $E_\lambda$ is then hypoelliptic.

Assume $E_\lambda$ a parametrix to a homogeneously hypoelliptic, constant coefficients differential operator $L_\lambda$, that is $L_\lambda I_{E_\lambda} = I - I_\gamma$ on $L^2$. Assume $Y_0 = N(L_\lambda^{-1})$, where $L_\lambda^{-1}$ is the Fredholm-inverse operator and $R(I_\gamma) = Y_0, N(I_\gamma) = X_0$. Further, $I_{E_\lambda} L_\lambda = I - I_\eta$ on $L^2$, such that $N(I_\eta) = R(L_\lambda)$, $R(I_\eta) = N(L_\lambda)$, then $E_\lambda$ works as a Fredholm-inverse operator to $L_\lambda$. Given $X_0$, $R(L_\lambda)$, by adding a regularizing operator if necessary, we can find $\gamma, \eta$ with these null-spaces. Since the operator $I_\gamma = I_\gamma Q$, with $Q$ regularizing, also $I_\gamma$ will be regularizing. An analogous argument, gives that also $I_\eta$ is regularizing. Thus, any parametrix to a homogeneously hypoelliptic operator, can be adjusted to a Fredholm-inverse operator.

## 21. Some remarks on the distribution parametrices

For a constant strength operator, extended with constant coefficients outside a compact set, we have seen that Levi's parametrix method, gives parametrices in

$\mathcal{D}'_{L^2}$. Assume $P_\lambda(D)$ a hypoelliptic operator with constant coefficients and $E_\lambda$ a parametrix to this operator. Thus

$$\| I_{P_\lambda E_\lambda}(\varphi) \|_{L^2} \leq \| \varphi \|_{L^2} + \| I_\gamma(\varphi) \|_{L^2}$$

and we can assume $\gamma \in C_0^\infty$. We have that $P_\lambda E_\lambda \varphi \in L^2$ and as we shall see, a "converse to Hölder's inequality" gives that $E_\lambda \in L^2$. Particularly, $E_\lambda : L^2 \to L^2$. If an operator has parametrix on the form of a tensor product with the Dirac measure, this will be in $\mathcal{D}'^{-m+1,n/2+1}_{L^2}$ and it maps $L^2 \to L^2$. More generally, any constant coefficients operator parametrix in $\mathcal{D}'^{l}_{L^2}$, for $k+ \mid \alpha \mid \geq n/2+1+m$, $\mid \alpha \mid \leq l$, and $k/2$ the order of the polynomial of growth for the "multiplier", corresponds to a bounded integral operator $L^2 \to L^2$. For a more general parametrix in $\mathcal{D}'^{s}_{L^2}$, corresponding to a variable coefficients operator $L_\lambda$, we have that $I_{E_\lambda}(\varphi) = Q(D)F$, where $Q(D)$ is a constant coefficients polynomial of order $s$ and $F \in L^2$. We then have that, if $L_\lambda(x,D)Q(D)F \in L^2$ and assuming the order of $Q$ larger than $n/2$, that is $s \geq n/2+1$, then

$$\| I_{E_\lambda}(\varphi) \|_{L^2} \leq C \| L_\lambda(x,D)F \|_{L^2} < \infty$$

Assume $E(\overline{x},y)$ corresponds to the conjugate with respect to $x$ in the kernel, that is if for instance $E = E_1 \otimes E_2$, we have $E(\overline{x},y) = \overline{E_1} \otimes E_2$. Let $E(\overline{x},\overline{y}) - E(x,y) = E(\overline{x},\overline{y}) - E(\overline{x},y) + E(\overline{x},y) - E(x,\overline{y}) + E(x,\overline{y}) - E(x,y) = \Sigma_1 + \Sigma_0 + \Sigma_2$. For a symmetric operator on $L^2$, we have $\Sigma_1 = \Sigma_2 = 0$, meaning that the operator is symmetric separately in the respective variables. For a symmetric operator acting on $\mathcal{D}'^F$, we have $I_{\Sigma_0} \sim \text{ Im } I_E$. If the operator is symmetric on $\mathcal{D}'$ (that is ${}^t I_E = I_E^*$), we have $\text{ Im } I_E = 0$, but this is usually not the case for homogeneously hypoelliptic operators. However, we always have that $I_E : \mathcal{D}' \to \mathcal{D}'^F + \mathrm{i}\mathcal{D}'^F$.

Assume $I_E^* = \overline{I}_E$ on $\mathcal{D}'^F$. Then, for $\varphi \in C_0^\infty$ real
$C_{\overline{I}_E} = \overline{I}_E \varphi - \varphi \overline{I}_E = \big[\varphi I_E\big]^* - \big[\varphi \overline{I}_E\big]$. This is regularizing on $\mathcal{D}'$. Thus, $-2i\varphi \text{ Im } I_E = \varphi(I_E^* - I_E) - C_{\overline{I}_E}$. By extending the right side with $\pm\varphi I$, we get a regularizing effect on $\mathcal{E}'$ as before. The difference between a homogeneously hypoelliptic and a hypoelliptic operator is that for the latter it is sufficient to consider the real part of the operator $P$, which means that we can assume $I_E^* = I_E$ on $\mathcal{D}'$ and this gives a regularizing effect for $\text{ Im } I_E$ on $\mathcal{D}'$.

**Proposition 21.0.2.** *Any constant coefficients, homogeneously hypoelliptic differential operator on $\mathcal{D}'$ is a hypoelliptic operator on $\mathcal{D}'^F$.*

Proof: Assume $P$ homogeneously hypoelliptic on $\mathcal{D}'$ with parametrix $E$. We have considered the operators

$$I_E : C^\infty \cap \mathcal{D}'^F \to C^\infty$$

meaning $P(D)E - I \in C^\infty$ over $\mathcal{D}'^F$

$$C_{I_E} : C^\infty \cap \mathcal{D}' \to C^\infty$$

If $P(D)u \in L^2_{loc}$ and $P(D)u \in C^\infty(\Omega)$ for all open sets $\Omega \subset \mathbf{R}^n$, then for any test function $\varphi$

$$\text{sing supp } \varphi u = \text{ sing supp } \Big( I_E \varphi P(D)u - C_{I_E} P(D)u \Big)$$

and we have $\varphi u \in C^\infty$. $\square$

## 22. Partially hypoelliptic operators considered as Fredholm operators

Also, if $\sigma$ arbitrary, that is $L_\lambda$ not necessarily homogeneously $L^2$- hypoelliptic, we can extend the definition of $E_\lambda$ with a regularizing term and we get a hypoelliptic action on, at least part of $L^2$. We now assume the operator $L_\lambda$ self-adjoint and partially self-adjoint.

If $E_\lambda \in \Phi$, is on the form $I - K$, there is a positive integer $N_0$, such that $N(E_\lambda^N) = N(E_\lambda^{N_0})$ for all $N \geq N_0$. We can rewrite $T_\lambda^{-1} = (I - \lambda E_\lambda)L_\lambda = L_{2\lambda}$ in $X_0 \bigoplus N(L_\lambda)$ and $T_\lambda^{-1} = L_\lambda(I - \lambda E_\lambda) = L_{2\lambda}$ on $R(L_\lambda) \bigoplus Y_0$. We get a new decomposition $Y = N(E_\lambda^{N_0}) \bigoplus N(E_\lambda^{N_0})^\perp$ and a corresponding decomposition of the operator $T$, $T = T_1 \bigoplus T_2$, with $T_1 = \sum_1^{N_0-1} \lambda^j E_\lambda^j$ and $T_2$ $L^2$-hypoelliptic. That is $E_\lambda : L^2 \to H^{s,t}$, where $s, t$ can be chosen arbitrary large, so on $N(E_\lambda^{N_0})^\perp \cap R(L_\lambda)^\perp$, $E_\lambda$ can be defined as regularizing. We can rewrite the first expression for $T_2^{-1}$, as $\lambda E_\lambda = I_\lambda - L_{2\lambda} E_\lambda$, where $I_\lambda$ denotes the identity operator in $R(L_\lambda)$. But $R(L_{2\lambda}) \subset R(L_\lambda)^\perp$ and we have that $E_\lambda$ is in fact hypoelliptic on $N(E_\lambda^{N_0})^\perp$. Thus, $\text{sing supp}_{L^2}(T_2\varphi) = \text{sing supp}_{L^2}(\varphi)$, for $\varphi \in L^2$ and for $u$ sufficiently regular, the integral $I_{T_2}(u)$, can be estimated in supremum norm without having to regularize the kernel.

We have seen that $E_{2\lambda}$ is $L^2$-hypoelliptic in the bands of ranges surrounding and including $R(L_{2\lambda})$, but since $N(E_{N\lambda}) = \{0\} \Rightarrow L_{N\lambda}$ $L^2$-hypoelliptic, it is has only $L^2$-action in the outer bands. Since $N(E_\lambda^{N_0})$ is a finite-dimensional space, normed with $L^2$-norm, nuclearity gives that this $L^2$-action can be defined by a kernel in $L^2$. For $f \in R(L_{2\lambda})$, we have the estimates, $\parallel E_{2\lambda} f \parallel_{L^2} \leq C \mid \lambda \mid^{-1} \parallel f \parallel_{L^2}$.

We can define a "(s,t)"-regularizing operator $C_{s,t} : L^2 \to H^{s,t}$, as $*'\mathcal{U}_{-s} *'' \mathcal{U}_{-t}$, for positive real numbers $s, t$. Let $E_{\lambda,t} = C_{0,t} E_\lambda$, where $E_\lambda$ is the parametrix constructed in $H^{0,-N}$, mentioned above. Then, for $t$ sufficiently large ($> N$), $E_{\lambda,t}$ is a parametrix in $L^2$, corresponding to a $L^2$-hypoelliptic operator. That is the condition that an operator $P_\lambda : H^{0,t} \to L^2$ has no derivatives in the $x''$-variables, means that the operator $P_{\lambda,-t}$ is hypoelliptic as an operator $L^2 \to L^2$. Thus $\text{sing supp}_{L^2}(P_{\lambda,-t} E_{\lambda,t}\varphi) = \text{sing supp}_{L^2}(E_{\lambda,t}\varphi) = \text{sing supp}_{L^2}(\varphi)$. Following the argument in the beginning of the section, this means that $E_{\lambda,t}$, can be extended with ($C^\infty$)-regularizing terms to $L^2$. Finally, $E_\lambda = (1 - \Delta_{y''})^{t/2} E_{\lambda,t}$, which means that the regularizing terms for $E_{\lambda,t}$, will be regularizing for $E_\lambda$ as well. Note that this does not necessarily mean that $P_\lambda$ is hypoelliptic, since its parametrix is not hypoelliptic on $R(P_\lambda)$. Thus, a parametrix on the form of a tensor-product with the Dirac-measure, can be used to extend the definition of $E_\lambda$, outside the range of the operator. On the range we can use the Fredholm-inverse,$P_\lambda^{-1}$ which gives a hypoelliptic action, for any differential operator $L_\lambda$.

On the bands, where $E_\lambda$ is regularizing, we have analytic dependence on $\lambda$, for $v_\lambda$ and this gives estimates like

$$\parallel E_\lambda^N \parallel_c \leq C_N \mid \lambda \mid^{-N} \qquad \lambda \text{ finite}$$

and particularly, $\parallel E_\lambda \parallel_c \leq C \mid \lambda \mid^{-N} \qquad \forall N$ and for $\lambda$ large, on these bands.

Note that the parametrix to $L_\lambda$ in $L^2$, can be written $E_\lambda = \bigoplus_{j=-\infty}^{\infty} E_\lambda P_{j\lambda}$, where $P_{j\lambda}$ is the projection on $R(L_{j\lambda})$. Since adding a compact operator to the

Fredholm-inverse, does not change the index or the form $I-K$, we let $E_\lambda$ be defined as $L_\lambda^{-1}+X$ on $R(L_\lambda)$ with $X \in C^\infty$, as $X' \in C^\infty$ on $R(L_\lambda)^\perp \cap N(E_\lambda^M)^\perp$, for some suitable $M$ and as a function in $L^2$ otherwise. Thus, this operator $L_\lambda$ is hypoelliptic on $N(E_\lambda^M)^\perp$. That is, for $\varphi \in L^2$,
sing $\text{supp}_{L^2}(\varphi) =$ sing $\text{supp}_{L^2}(L_\lambda E_\lambda \varphi) \subset$ sing $\text{supp}_{L^2}(E_\lambda \varphi)$, further
sing $\text{supp}_{L^2}(L_\lambda E_\lambda \varphi) =$ sing $\text{supp}_{L^2}(E_\lambda L_\lambda \varphi)$ and the result follows.

## 23. Hypoellipticity in the $\lambda$-infinity

Assume $E_\lambda$ the parametrix to an operator with constant coefficients $L_\lambda$. The parametrix setting gives that, if $L_\lambda$ is self-adjoint, $E_\lambda$ has Fredholm-index 0, and that $\lim_{n\to\infty}$ dim $N(E_\lambda^n) = \lim_{n\to\infty}$ codim $N(E_\lambda^n) < \infty$. If there is an index $N_0$, such that $E_{N\lambda} = I-K$, for $N \geq N_0$ and for $K$ regularizing, then $E_{M\lambda} = I-K'$ with $K'$ regularizing, for all $\mid M \mid \leq N_0$. According the previous paragraph, $E_{N\lambda}$ will be on this form as $\mid N \mid \to \infty$. More precisely, assume for instance, $(\delta_x - E_{2\lambda}) \in \mathcal{R}$, where $\mathcal{R}$ is the set of regularizing operators in $L^2$. Further that $(\delta_x - E_{2\lambda+1}) \in \mathcal{R}$, which particularly means that $E_{2\lambda+1}$ is pseudo local. The first condition means that $L_{2\lambda+1}E_{2\lambda} \in \mathcal{R}$. Finally, $E_{2\lambda+1}L_{2\lambda+1}E_{2\lambda}(\delta_x - E_\lambda) \in \mathcal{R}$, which through the second condition means that $(\delta_x - E_\lambda) \in \mathcal{R}$. We could say, that operators on the form $I-K$ in $L^2$, constitute a radical subsystem among $L^2$-hypoelliptic operators. Note that if the kernel $N(E_\lambda) = Y_0 \neq \{0\}$, the decomposition $L^2 = R(L_\lambda) \oplus Y_0$ indicates that $L_\lambda$ still can not be considered as hypoelliptic on $L^2$.

## 24. Asymptotically $L^2$-hypoelliptic operators

We now claim that $E_\lambda^2$ is a parametrix to the operator $L_\lambda^2$. That is
$E_\lambda^2 = (E_\lambda P_\lambda + E_\lambda(P_\lambda^\perp))^2 = (E_\lambda P_\lambda)^2 + (E_\lambda P_\lambda^\perp)^2$. If $E_\lambda$ is $L^2$-hypoelliptic, then the same must hold for $E_\lambda^2$. We make the following definition,

**Definition 24.0.1.** *Assume $E_\lambda$ a parametrix in $L^2$, to the operator $L_\lambda$ and that $E_\lambda$ is $L^2$-hypoelliptic on $N(E_\lambda^{N_0})^\perp$ with $N_0$ chosen as the smallest positive integer, such that the null space remains stable. We then say that $L_\lambda$ is hypoelliptic on $L^2$, if $N_0 = 1$ and that it is asymptotically ($N_0$-) hypoelliptic, if $N_0 > 1$.*

We claim that if $L_\lambda$ is asymptotically $N_0$-hypoelliptic, then $L_\lambda^{N_0}$ is hypoelliptic in $L^2$. Assume $E_{\lambda,N_0}$ the usual $L^2$-parametrix to $L_\lambda^{N_0}$. We first have to prove, that $N(E_{\lambda,N_0}^{N_1}) = N(E_\lambda^{N_0}) \Rightarrow N_1 = 1$. We have that $E_{\lambda,N_0}L_\lambda^{N_0} = L_\lambda^{N_0}E_{\lambda,N_0} = \delta_x - \gamma$, for some $\gamma \in C^\infty$ and $\gamma = 0$ on $N(E_\lambda^{N_0})$. Thus $N(E_\lambda^{N_0}) = N(E_{\lambda,N_0})$, that is $N_1 = 1$ and $L_\lambda^{N_0}$ is hypoelliptic on $L^2$.

Note that given a parametrix $E_{\lambda,N_0}$ to a $L^2$-hypoelliptic operator $L_\lambda^{N_0}$, if $N(E_{\lambda,N_0}) \neq \{0\}$, we can always add to $E_{\lambda,N_0}$, a solution to the homogeneous equation , $H$, non-zero on $N(E_{\lambda,N_0})$, so that $N(E_{\lambda,N_0}+H)$ is trivial. We can now assume $N(E_{\lambda,N_0}) = \{0\}$ and get $E_{\lambda,N_0} - E_\lambda^{N_0} \in C^\infty$. That is the parametrix to $L_\lambda^{N_0}$, has a regularizing action on $N(E_\lambda^{N_0})$.

The fact that $L_\lambda^{N_0}$, is $L^2$-hypoelliptic, renders a spectral kernel in $C^\infty$. The relation between the spectral kernels corresponding the operator and the iterated operator, gives the spectral kernel corresponding to $L_\lambda$ in $L^2$. This however, does not imply that it is in $C^\infty$, on $N(E_\lambda^{N_0})$.

Assume $U_\lambda = \{\xi \in \mathbf{R}^\nu, \mid \xi'' \mid \leq \mid \lambda \mid\}$ , then $L^2(\mathbf{R}^\nu) = L^2(U_\lambda) \bigoplus L^2(\mathbf{R}^\nu \setminus U_\lambda)$, where $L^2 \ni f = f' + f''$, and supp $\hat{f}' \subset U_\lambda$. Assuming the frozen operator, $L_\lambda$,

hypoelliptic in $x'$, we can assume the corresponding parametrix, $E_\lambda = E'_\lambda \otimes E''_\lambda$, adjusted to $N(E'_\lambda) = \{0\}$. The operator $E''_\lambda$, is well defined after adjusting the $L^2$-element, $\hat{f}''$ to a compact set. Thus $N(E_\lambda) \subset L^2(\mathbf{R}^\nu \backslash U_\lambda)$. It is for the variable coefficient case, sufficient to study operators on tensor form $L^\Sigma_\lambda$. This only involves an operation on $E'_\lambda$, which does not effect $N(E'_\lambda)$.

## 25. Some remarks on Weyl's criterion

- According to L. Schwartz, a condition equivalent with hypoellipticity for a differential operator $P$, is that $P$ and ${}^tP$ (the transposed operator) have parametrices, that are very regular. For the fundamental solution, we obviously have that Weyl's lemma implies that the fundamental solution (kernel) is very regular. For the opposite implication, there are counter examples in the variable coefficients case, for example the following differential operator (by Mizohata cf.[14],[22])

$$P = \frac{\delta}{\delta x_1} + i x_1^h \frac{\delta}{\delta x_2} \qquad h \text{ integer}$$

with fundamental solution $e(x,y) = \frac{1}{2\pi}\big(x_1^{h+1}/(h+1) + ix_2 - y_1^{h+1}/(h+1) - iy_2\big)^{-1}$. When $h$ is odd, $P$ is not hypoelliptic, since one solution to the equation $Pu = 0$ is $u(x) = \big(x_1^{h+1}/(h+1) + ix_2\big)^{-1}$. Finally, we wish to remark that the condition that the differential operator $P(D)$ is dependent on some variables in space, is essential for the opposite implication to hold. A trivial counterexample is the identity operator. This operator has the property of microlocal hypoellipticity, it is a hypoelliptic pseudo differential operator, but as we shall see, it is not a hypoelliptic differential operator.

We have obviously, in the constant coefficients case, that if Weyl's criterion is to be written sing supp $P(D)u =$ sing supp $u$ for all $u \in \mathcal{D}'(\Omega)$ and every $\Omega \subset \mathbf{R}^n$, that we must consider the situation outside ${\mathcal{D}'}^F$ and the Sobolev-spaces. Assume $P$ with $\sigma > 0$ but not hypoelliptic ( that is not self-adjoint ) and consider

$$\| (P^* - P)u \| \leq C \| Pu \| \qquad u \in H_K^{0,0} \tag{35}$$

for some constant $C$. This criterion can always be satisfied. Using $P^* - \overline{P} + \overline{P} - P = P^* - \overline{P} - 2i \text{ Im } P$, if $P$ is localized with $\varphi \in C_0^\infty$, real and such that $C_{\overline{P},\varphi} \neq 0$ on a compact set, we have $C_{\overline{P},\varphi} - 2i\varphi \text{ Im } P = \big[\varphi P\big]^* - \big[\varphi P\big]$. Since, according to (35), $P^* - P \prec\prec P$, if $P$ has $\sigma > 0$ and if we assume $\overline{P} = P^*$ hypoelliptic, we must have Im $P \prec\prec P$. Consider, for example $P + iP$, with $P$ hypoelliptic and real, then this is a homogeneously hypoelliptic operator, but not necessarily a hypoelliptic operator in $\mathcal{D}'$. With the additional condition that the operator is self-adjoint, we must however have that it is hypoelliptic.

Consider now parametrices $E$ to partially hypoelliptic operators. We have seen that $E^N : \mathcal{D}' \to {\mathcal{D}'}^F$, for some $N$, but this does not mean that $E : \mathcal{D}' \to {\mathcal{D}'}^F$. For example, $T^2 = (\gamma - i\delta_x)^2 = \gamma^2 - \delta_x - 2i\gamma : \mathcal{D}' \to {\mathcal{D}'}^F$, but $T : \mathcal{D}' \to \mathcal{D}'$, particularly Im $T : \mathcal{D}' \to \mathcal{D}'$. Further, $\mid Tu \mid^2 \in {\mathcal{D}'}^F$, $u \in \mathcal{D}'$ does not imply that $Tu \in {\mathcal{D}'}^F$, for instance $u$ real, if $w = (\gamma + i)u(\gamma - i)u$ and $(\gamma + 1)u(\gamma - 1)u = -v$. Using that $v \in {\mathcal{D}'}^F$ and

$w - v \in \mathcal{D}'^F$, we see that $w \in \mathcal{D}'^F$. We have already seen that geometric ideals have the property that the ideal is its complexification if and only if it is radical and since the class of parametrices to partially hypoelliptic operators are associated to geometric ideals, we get the same type of behavior for this class. Note that a necessary condition for hypoellipticity in $\mathcal{D}'$ is thus that Im $P \prec\prec P$ and this condition will be satisfied by iteration, that is $C_P^N \prec\prec P^N$ from some $N$, implies Im $P^N \prec\prec P^N$. More precisely, assume $(,)$ a complex scalar product over a Hilbert space $H$. Thus, Re $(x, y) =$ Re$(ix, iy)$ $\quad x, y \in H$ and $(x, y) =$ Re $(x, y) - i$ Re $(ix, y)$. Then, for a constant coefficients differential operator $P = P_1 + iP_2$,

$$\text{Im } \big[(P^2\varphi, \varphi) - (P^*P\varphi, \varphi)\big] = 2i \text{ Re } (P_1\varphi, P_2\varphi)$$

for $\varphi \in C_0^\infty$ arbitrary. If $P^* = P$, we must have

(36) $$i \text{ Re } (P_1\varphi, P_2\varphi) = 0$$

Conversely, if (36) then $(P - P^*)\perp P^*$ over $C_0^\infty$. Thus, if $E$ is the projection $R(P^*)^\perp \rightarrow N(P^*)$, then $P^*E(P - P^*)\varphi = 0$ for $\varphi \in N(P^*)^\perp$. Particularly, if $\varphi$ is chosen as the mollifier (cf. the proof of Prop. 25.2), we have $\parallel P^2\varphi_n \parallel \rightarrow \mid P(\xi) \mid^2$ as $n \rightarrow \infty$.

**Lemma 25.1.** *Assume $P$ and $Q$ constant coefficients differential operators such that $P \prec Q$, $Q$ hypoelliptic and $(P\varphi, Q\varphi) = 0$ for all $\varphi \in N(Q)^\perp$. Then $P \prec\prec Q$.*

Proof: For a Fredholm inverse $E$ to $Q$ we have that $PE : L^2 \rightarrow H^\sigma$ for a $\sigma \geq 0$. For an appropriate $\varphi$, Hölder's inequality gives that $\mid \xi \mid^\sigma P(\xi)\widehat{E} \rightarrow 0$ as $\mid \xi \mid \rightarrow \infty$, which is interpreted as $P \prec\prec Q$. $\square$

Assume now $P_2 \prec\prec P_1$, we can then find an entire $f$ with $P_2 = fP_1$ and

$$(P_1\varphi, P_2\varphi) = \int f \mid P_1\varphi \mid^2 d\xi$$

with $\mid f \mid \rightarrow 0$ as $\mid \xi \mid \rightarrow \infty$. If we assume $f$ adjusted to $f'$ with support outside a ball containing the origin, we could say $(P_2\varphi, P_1\varphi) \sim 0$. For all $\varphi \in C_0^\infty$ we have $P_2\varphi \in C_0^\infty$ and this means that there exists a $\gamma_\varphi$ regularizing, such that $P_2(I - \gamma_\varphi)\varphi = 0$. Further, $(P_2P_1E_1^\varphi\varphi, E_1^\varphi\varphi) = 0$ and $P_1E_1^\varphi = I - \gamma_\varphi$. Assume $P_1$ hypoelliptic, then for all $\psi \in C_0^\infty$, there is a $\varphi \in C_0^\infty$, such that $E_1^\psi\varphi = \psi$ ($E_1^\psi$ is chosen according to the support of $\psi$). If we choose $\psi$ so that $\parallel \psi \parallel = 1$ in the mollifier, we have that $\parallel P^2\varphi_n \parallel \rightarrow \mid P(\xi) \mid^2$ as $n \rightarrow \infty$. If $P_1$ is only partially hypoelliptic, we can consider $Q_N = P_1^N + iP_2^N$, such that Im $\big[(Q_N^2\varphi, \varphi) - (Q_N^*Q_N\varphi, \varphi)\big] = 2i$ Re $(P_1^N\varphi, P_2^N\varphi)$ and the above argument gives that $\parallel Q_N^2\varphi_n \parallel \rightarrow \mid Q_N(\xi) \mid^2$ as $n \rightarrow \infty$.

Assume $E_1 \in C^\infty(\mathbf{R}^\nu \setminus 0)$ a parametrix, that is with $\gamma \in \mathcal{D}$, such that $\gamma = 1$ in a neighborhood 0,

$$P(D) * (\gamma E_1 * \varphi) = \zeta * \varphi + \varphi \qquad \zeta \in \mathcal{D}$$

which means

$$\varphi = \gamma E_1 * P(D)\varphi - \zeta * \varphi$$

This is sufficient to conclude that sing supp $\varphi \subset$ sing supp $P(D)\varphi$. If we use Leibniz' formula to construct a parametrix $\gamma E_1$, with $\gamma$ as before, that

is,

$$P(D) * (\gamma E_1) = \gamma P(D) * E_1 + \zeta = \delta_0 + \zeta \qquad \zeta \in \mathcal{D}$$

where $\zeta = 0$ in a neighborhood 0, the same conclusion holds, unless $E_1$ has no support in the complement of the origin ($\zeta \equiv 0$). In this case the corresponding operator is not necessarily a hypoelliptic differential operator.

Note that the fundamental solution to a constant coefficients, hypoelliptic differential operator, is very regular in ${\mathcal{D}'}^F$, but it is not hypoelliptic in $\mathcal{D}'$ according to the argument above, since this would mean that the Dirac measure is hypoelliptic.

- The regularity behavior for PHE operators, can be expressed using Sobolev spaces, in the following way:

We have seen that a fundamental solution $g_\lambda$ with singularity in 0, to the operator $P(D_{x'}) - \lambda$, can be given on the form $g_\lambda = {\mathcal{F}'}^{-1}\Big(\frac{1}{\mathrm{P}(\xi') - \lambda}\Big) \otimes \delta_0$. We then have, for $\psi \in C_0^\infty$ so that $\psi = 1$ in a neighborhood $U$ in $\mathbf{R}^\nu$ of 0,

$$\psi u = g_\lambda * \Big(\psi L_\lambda u + (L_\lambda \psi - \psi L_\lambda) u - QR(\psi u)\Big) = g_\lambda * \Big(\psi L_\lambda u + B_\lambda u - R'(\psi u)\Big)$$

We have, $B_\lambda = 0$ in $U$ and $R'(\psi u) = \psi R' u + R'' u$ with $R'' u = 0$ in $U$. According to Proposition 1.9, we have $\psi L_\lambda u = \psi P_\lambda u + \psi R' u$, so $\psi u = g_\lambda * \psi P_\lambda u$. Is $P_\lambda$ hypoelliptic? Since $C^\infty(U) = \cap_{s,t} H^{s,t}_{loc}(U)$, it would be sufficient to prove $\parallel \psi u \parallel_{s,t} \leq C \parallel \psi P_\lambda(D_{x'}) u \parallel_{s,t}$, given arbitrary real numbers $s, t$. Let $f_\sigma(D_{x'}, D_{x''}) = \Big(P(D_{x'}) - \lambda\Big)\Big(1 - \Delta_{x''}\Big)^\sigma$, for a non-negative number $\sigma$. We then see that $F_\sigma = \mathcal{F}\ (f_\sigma \delta_0)$ is a weight function, defining a Banach space $H_{F_\sigma}$ through the norm $\parallel u \parallel^2_{F_\sigma}$ $= \int \mid \mathcal{F}\ (u)(\xi) \mid^2 F_\sigma(\xi) d\xi$ and $F_\sigma^{-1}$ gives the antidual space to $H_{F_\sigma}$. According to [[12], Lemma I.2.2]:

**Proposition 25.1.** *Given two weight functions $h, f$, if*

$$\lim_{\xi \to \infty} \frac{h(\xi)}{f(\xi)} = 0$$

*then for a positive constant C, $\parallel \cdot \parallel_h \leq C \parallel \cdot \parallel_f$.*

**Proposition 25.2.** *Assume $M(D_{x'}), N(D_{x'})$ constant coefficients operators, where $M$ is assumed hypoelliptic over $\mathbf{R}^n$, then $N$ is strictly weaker than $M$ if and only if there is a positive number $\sigma$, such that*

$$\parallel N(D_{x'}) f \parallel_{s+\sigma,t} \leq C_K \parallel M(D_{x'}) f \parallel_{s,t} \qquad \textit{for all } f \textit{ in } \mathcal{E}'(K) \tag{37}$$

*Here $s$ and $t$ are arbitrary real numbers.*

Proof: The implication $N \prec\prec M \Rightarrow$ (37), is proved in [15]. For the opposite implication, it is sufficient to consider the case $s = t = 0$. We use the mollifier (cf.[24]) $\varphi_k(x) = \mathrm{e}^{i\xi \cdot x} \psi(x/k)/k^{n/2}$, $k = 1, 2, \ldots$, where $\psi \in C_0^\infty$, such that $\parallel \psi \parallel_{0,0} = 1$. Using that the right side norm in (37) is equivalent to $\parallel \cdot \parallel_{H^{0,0}_K(M)}$, we see that

$$\parallel N(D_{x'}) \varphi_k \parallel_{\sigma,0} \leq C_K \parallel \varphi_k \parallel_{H^{0,0}_K(M)}$$

Taking the limit as $k \to \infty$, we have that

$$\frac{(1 + \mid \xi' \mid^2)^\sigma \mid N(\xi') \mid}{1 + \mid M(\xi') \mid} \qquad \text{bounded for } \xi' \in \mathbf{R}^n$$

This is sufficient to conclude $N \prec\prec M$. □

Proposition 25.2 gives particularly, that a necessary condition for hypoellipticity in $\mathcal{D}'$, for an operator $P(D)$, is that $Id \prec\prec P(D)$, this means in our case, $Id \prec\prec f_\sigma(D)$, for all $\sigma \geq 0$. Proposition 25.1 gives that $\parallel f_\sigma^{-1} * u \parallel_{s,t} \leq C \parallel u \parallel_{s,t}$, for $u \in H^{s,t}$. So,

$$\parallel \psi u \parallel_{s,t} = \parallel g_\lambda * \psi P_\lambda(D_{x'})u \parallel_{s,t} \leq C \parallel \psi P_\lambda(D_{x'})u \parallel_{s,t+\sigma}$$

but in the case of a partially hypoelliptic operator, we have to assume $\sigma > 0$ and if $P_\lambda(D_{x'})u \in C^\infty(U)$, we do not necessarily have that $u \in C^\infty(U)$. Assume $U$ a neighborhood of $x$, $\psi$ a test function with support away from 0, further that $u$ is a fundamental solution to $L_\lambda(D)$, we then have $\parallel \psi u \parallel_{s,t} = \parallel g_\lambda * \Big(\psi L_\lambda(D)u + B_\lambda u + R'(\psi u)\Big) \parallel_{s,t}$. So on $U$, a fundamental solution to a partially hypoelliptic operator cannot have better regularity properties than $g_\lambda$. Note that for the identity operator, the necessary condition for hypoellipticity in $\mathcal{D}'$, is not satisfied.

## 26. Some results on the spectral kernel for partially hypoelliptic operators

26.1. **Partial regularity for the spectral kernel.** In this section, we follow the arguments of Nilsson [16]. Assume $H = L^2(\mathbf{R}^\nu)$ is our Hilbert space. For the spectral family $\{E(\lambda)\}$, associated to a realization, $\mathcal{A}_L$, corresponding to the formally self-adjoint operator $L(x, D_x)$ and for $(\lambda_1, \lambda_2] \subset \mathbf{R}$, we define the operators $E(\lambda_1, \lambda_2) = E(\lambda_2) - E(\lambda_1)$, as projections on the subspace of $H$, $H(\lambda_1, \lambda_2) = H(\lambda_2) \ominus H(\lambda_1)$ (the minus sign denotes the orthogonal complement, of $H(\lambda_1)$ in $H(\lambda_2)$). For a given closed interval $I$ and a corresponding partition of finitely many subintervals $\{I_j\}_{j=1}^N$, each of length $\leq \epsilon$, we can write $H_I = \bigoplus_{j=1}^N H_{I_j}$, where each of the subspaces is invariant for $\mathcal{A}_L$. For $\lambda_j \in I_j$ and for every $x \in H_{I_j}$, we have

$$\parallel (\mathcal{A}_L - \lambda_j)x \parallel = \parallel ((\lambda - \lambda_j)\chi_{I_j})(\mathcal{A}_L)x \parallel \leq \sup_I \mid (\lambda - \lambda_j)\chi_{I_j}(\lambda) \mid \parallel x \parallel \leq \epsilon \parallel x \parallel$$

that is $x$ is an $\epsilon-$approximative eigenvector to $\mathcal{A}_L$ (cf.[20]).

We note that the spectral family or orthogonal spectral resolution, uniquely determined by the operator $\mathcal{A}_L$, is a regular countably additive spectral measure, non-decreasing and such that $E(\lambda) \to 0$, as $\lambda \to -\infty$ and $E(\lambda) \to I$, as $\lambda \to +\infty$. According to the spectral theorem, we have $\mathcal{A}_L u = \int_{\mathbf{R}} \lambda dE(\lambda)u$ with strong convergence, for $u$ in the domain of $\mathcal{A}_L$.

**The spectral resolution.** We can show (analogously to [15] Theorem 3), using Proposition 3.2.1, for $kr > n/2 + l$, $\mid \alpha \mid \leq l$ and for a compact set $K$, there is an integer $N$ such that

$$\sup_K \mid D^\alpha f *'' \mathcal{U}_{-2N} \mid \leq C_K \parallel f \parallel_{H_K^{s,-N}(P^r)} \tag{38}$$

Also, if $\tilde{f} = f *'' \varphi$, $\varphi$ a test function with support in neighborhood 0

$$\sup_K \mid D^\alpha \tilde{f}(y) \mid \leq C(K)\big( \parallel \mathcal{A}_{P^r} f \parallel_H + \parallel f \parallel_H \big) \quad f \in H(\lambda_1, \lambda_2) \tag{39}$$

The following result follows immediately from [8,II Theorem 10.4.8], where our additional condition, is due to the fact that we let $\alpha \to \infty$

**Lemma 26.1.1.** *Given a hypoelliptic, constant coefficients operator $P$, adding $\alpha Q$, where $\alpha$ is a complex constant, and $Q$ a strictly weaker, constant coefficients operator, such that $\mid P(\xi)+\alpha Q(\xi) \mid \neq 0$ for $\mid \xi \mid$ sufficiently large, gives an operator equivalent with $P$*

**The constant coefficients case.** In section 13 , we constructed a fundamental solution with singularity in $x$, $h_\lambda$, to the constant coefficients operator $L_\lambda(D_y)=L(D_y)-\lambda=P_\lambda+R$, with singular support on $\Sigma_x=\{z; z''=x''\}$. For a suitable test function $\zeta \in C_0^\infty(W)$, $\zeta=1$ on $\Sigma_x \cup F_2$, we can construct the parametrix $G_\lambda=\zeta h_\lambda$ to $L_\lambda$, as $L_\lambda(\zeta h_\lambda)=\delta_x-\eta_\lambda$, where $\eta_\lambda$ is in $C^\infty$. This gives a representation formula (similar to [15]), for $u$ sufficiently regular and for concentric spheres $F_4 \subset F_3 \subset F_2 \subset F_1 \subset W$, where $W$ is a neighborhood of the singularity $x$. For a $\psi \in C_0^\infty$, such that $\psi=1$ on $F_2$, we have for x $\in F_2$

$$u(x)=\int_{F_1\setminus F_2} \big(\overline{B_\lambda(x,y)}+\overline{\eta_\lambda(x,y)}\big)u(y)dy+ \tag{40}$$
$$\int_{F_1} \overline{\psi(y)G_\lambda(x,y)}L_\lambda(D_y)u(y)dy$$

where $B_\lambda(x,y)=L_\lambda(D_y)(1-\psi(y))G_\lambda(x,y)$. According to the proof of Proposition 19.0.2.3, $\mid B_\lambda(x',y') \mid \leq C\exp(-\kappa \mid \lambda \mid^b)$, for $x \in F_3$, as $\lambda \to -\infty$ and the same estimate holds for $B_{-\lambda}$, as $\lambda \to \infty$, on compact sets in $\mathbf{R}^n$.

**Notation 26.1.2.** *We write $B_\lambda(x,y)=O(1)\exp(-\kappa \mid \lambda \mid^b)\otimes \delta_{x''}$, meaning $\mid B_\lambda *_x'' \varphi *_y \psi \mid \leq C\exp(-\kappa \mid \lambda \mid^b) \mid \varphi(x'')\psi(y'') \mid$, for $\varphi \otimes \psi \in C_0^\infty(\mathbf{R}^m \times \mathbf{R}^m)$.*

The same estimate holds for $\eta_\lambda$, since it only involves derivatives of $g_\lambda$ and of a test function. Further $G_\lambda^{(0,\beta')}=O(1) \mid \lambda \mid^{-c} \otimes \delta_{x''}$, as $\lambda \to -\infty$, uniformly on $\mathbf{R}^\nu \times \mathbf{R}^\nu$. Let's localize $u$ with a $\phi \in C_0^\infty$ $(\mathbf{R}^\nu)$, $\phi=1$ in $F_3$ and regularize with $\varphi \in C_0^\infty(\mathbf{R}^m)$ with support in a neighborhood of 0. We then have, for $\widetilde{u}_\delta=u *'' \varphi_\delta$, $\parallel \varphi_\delta \parallel_{L^1}=1$,

$$\parallel \widetilde{u}_\delta \parallel_{H(F_3)} \leq Ce^{-\kappa|\lambda|^b} \parallel u \parallel_{H(F_1\setminus F_3)} +C' \mid \lambda \mid^{-c} \parallel L_{-\lambda}u \parallel_{H(F_3)} \qquad \text{as } \lambda \to \infty \tag{41}$$

For finite $\lambda$, Lemma 26.1.1 gives that $L-\lambda \sim_{x'} L+\lambda$, so this estimate holds also for $L_\lambda$ with $\lambda$ large and positive, if we can prove that the equivalence constant is independent of $\lambda$. Note that we can assume $u$ with support contained in a bounded domain, which means that the equivalence implies $\parallel L_{-\lambda}u \parallel_H \leq C \parallel L_\lambda u \parallel_H$, for a constant independent of $u$ and $\lambda$. For the independence, we need the following result, that can be found in [2].

**Lemma 26.1.3.** *For constant coefficients operators $Q, M$, such that $M$ hypoelliptic and $Q \prec\prec M$, there are positive constants $C$ and $k$, such that*

$$\mid Q(\xi') \mid \leq C\tau^{-k}(1+ \mid \xi' \mid)^{-k}(\tau+ \mid M(\xi') \mid)$$

*for every $\xi'$ in $\mathbf{R}^n$ and $\tau \geq 1$ real.*

First, let's denote $L^{-2}=\mid L \mid^2+\mid \lambda \mid^2$ where $L$ is regarded as a polynomial over $\mathbf{R}^n$. Lemma 26.1.3 gives that $1/L^- \leq C/\mid \lambda \mid^\sigma$, for some positive number $\sigma$ and with the constant $C$ independent of $\lambda$. If $L^- \in \mathcal{H}_\sigma$ and $\sigma \geq 1$, $\mid \lambda \mid / \mid L \mid \leq C$, for every large $\lambda$. Otherwise, this result holds for the iterated operator $L^r$ and we have $L^{-,(r)} \sim_{x'} L^r$, for large $\lambda$. The same result gives also that $(L+\lambda)^r \sim_{x'} L^{-,(r)}$, for any large $\lambda$. We have the following result,

**Proposition 26.1.4.** *Given a hypoelliptic constant coefficients operator $P$, assume $P \in \mathcal{H}_\sigma$, for $\sigma \geq 1$. Then, for $\lambda$ complex*

$$P \pm \lambda \in \mathcal{H}_\sigma \qquad \mid \lambda \mid \to \infty$$

We have a condition that the type operator $M \in \mathcal{H}_\sigma$, for $\sigma > n$, which means that the operator need not be iterated any further, in order to be partially hypoelliptic independently of $\lambda$.

For $f \in H(\lambda - \epsilon, \lambda)$, $\epsilon > 0$, we have that f is an $\epsilon$-approximative eigenvector to L and $\lambda$, so $\parallel L_\lambda(D)f \parallel_H \leq \epsilon \parallel f \parallel_H$ and using (41)

$$\parallel \widetilde{f}_\delta \parallel_H \leq C\big(\exp(-\kappa \mid \lambda \mid^b) + \mid \lambda \mid^{-c} \epsilon\big) \parallel f \parallel_H$$

where $\lambda$ is assumed large and positive. The constant $C$ is not dependent on $u$, $\lambda$ or on the test functions used in the regularization.

For $f \in H(\lambda - k\epsilon, \lambda)$, k a positive integer, we can write $f = \sum_1^k f_j$, where $f_j \in H(\lambda - j\epsilon, \lambda - (j-1)\epsilon)$, so by selecting $\epsilon$ and $k$ in a suitable way, we get

$$\parallel \widetilde{f}_\delta \parallel_H \leq C \exp(-\kappa \mid \lambda \mid^b) \parallel f \parallel_H$$

where the constant $C$ is independent of $f$,$\lambda$ and the regularization and $\lambda$ is assumed large and positive.

For $f \in H(\lambda - 1, \lambda)$, we also have $\mathcal{A}_{P^r} f \in H(\lambda - 1, \lambda)$ so

$$\parallel \widetilde{\mathcal{A}_{P^r} f}_\delta \parallel_H \leq C_r \exp(-\kappa \mid \lambda \mid^b) \parallel f \parallel_H$$

Thus for $\mid \alpha \mid \leq l$ and kr > n/2 + 1

$$\sup_{\overline{F_4}} \mid D^\alpha f *'' \varphi(y) \mid \leq C(\alpha) \exp(-\kappa \mid \lambda \mid^b) \parallel f \parallel_H$$

where $F_4$ is a sphere with center $x_0$, $\varphi$ a test function with support in a sufficiently small nbhd of 0 and $C$ is independent of $f$,$\lambda$ and the regularization, but not of $F_4$. Here $\lambda$ is assumed large and positive. Finally, this result follows for a general compact set $K$, by the Heine-Borel theorem.

**The variable coefficients case.** Let's now consider the variable coefficients case. In section 13, we constructed a parametrix also to the variable coefficients operator $L_\lambda(y, D_y)\ F_\lambda = \delta_x - \gamma_\lambda$, with $\gamma_\lambda \in C^\infty$, where we have adjusted the singularity to $x$, such that $P_\lambda^\Sigma K_\lambda^\Sigma = \delta_x$. We get a representation formula similar to (41), where we assume $W$ a nbhd of $x_0$, in which the operator is partially formally hypoelliptic, $L_\lambda F_\lambda = \delta_x + \gamma_\lambda$ and $B_\lambda(x, y) = L_\lambda(y, D_y)(1 - \psi(y))F_\lambda(x, y)$ with support on $F_1 \setminus F_2$. To simplify the calculations we use a representation

$$u *'' \varphi_\delta(x) = I_{\overline{\psi B_\lambda^\delta}}(u)(x) + I_{\overline{\psi F_\lambda^\delta}}(L_\lambda(\psi u))(x) - I_{\overline{\psi \gamma_\lambda^\delta}}(u)(x)$$

where $B_\lambda^\delta = B_\lambda *'' \varphi_\delta$ and analogously for $\gamma_\lambda^\delta$. In order to produce an estimate like (41), we need a fine estimate of $\gamma_\lambda$. Using [15] (Cor. 2 to Prop 2.1) and Proposition 25.2, we can give the following result

**Lemma 26.1.5.** *Given a variable coefficients operator $P$, with coefficients in $C^\infty(\mathbf{R}^\nu)$ and $= 0$ on $\Sigma_x = \{(y', y''); y'' = x''\}$, we have*

$$\parallel P(y, D_y)T \parallel_{s+\sigma,-N} \leq \epsilon \parallel M(D_{y'})T \parallel_{s,-N'} \qquad M(D_{x'})T \in H_K^{s,-N'}$$

*where $s$ is a real number, $N, N'$ positive integers and $\sigma$ a real number that can be chosen as positive if $P \prec\prec_{x'} M$ and as zero if $P \sim_{x'} M$. Finally $\epsilon$ can be chosen arbitrarily small as the support for $T \to \Sigma_x$*

**Remark** : The set $\Sigma_x$ can be chosen in different ways, but assuming $E_\lambda = K_\lambda^\Sigma + \sum_j E_{\lambda,j} \otimes \delta_{x''}$, where $E_{\lambda,j}$ is an arbitrary solution to the homogeneous equation, $\Sigma_x$ according to the Lemma, seems to be the natural choice.

In section 13, we saw that the remainder corresponding to the parametrix, is on the form of operators $C_\lambda, D_\lambda$ with coefficients vanishing on $\Sigma_x$, acting on $E_\lambda$, that is $\gamma_\lambda = C_\lambda E_\lambda + D_\lambda E_\lambda$, where $C_\lambda \sim_{x'} M$ and $D_\lambda \prec\prec_{x'} M$. If we mollify $E_\lambda$ appropriately, we have $E_\lambda^\delta = E_\lambda *'' \varphi_\delta$ with supp $E_\lambda^\delta \to \Sigma_x$, as $\delta \to 0$. We then have the following estimate of the remainder term, for $\lambda$ sufficiently large and for some positive constant $c$,

$$\| \int \overline{\phi\psi\gamma_\lambda^\delta(x,y)} u(y)dy \|_H \leq \epsilon_1 \mid \lambda \mid^{-c} \| \phi \|_H \| \psi \|_H \| u \|_H \tag{42}$$

where we have used Lemma 26.1.5, Leibniz' formula, Cauchy-Schwarz' inequality and the estimate in Prop. 19.0.2. Here, $\epsilon_1$ is dependent on the value of the coefficients corresponding to $L_\lambda$ in a nbhd of $\Sigma_x$ and $\epsilon_1 \to 0$ as $\delta \to 0$. Further, $\epsilon_1$ is dependent on the mollifier, but it is not dependent on the support of $u$.

An argument similar to the constant coefficient case, gives for $\mid \alpha \mid \leq l$ and kr $> n/2 + l$,

$$\sup_K \mid D^\alpha f *'' \varphi_\delta(y) \mid \leq C(\alpha,\delta) \exp(-\kappa \mid \lambda \mid^b) \parallel f \parallel_H$$

and still, the constant $C$, is dependent on the mollifier and on the compact set $K$, but not on $f$ or $\lambda$.

**Remark** (1): The right hand side in (39) can be used as a definition of a norm. We prefer in this case to work with the Hilbert space $H_K^{0,-N}$, $K$ a compact set in $\Omega$ and we let $\mid f \mid_{r,N} = (\parallel f \parallel_{H_K^{0,-N}} + \parallel P^r f \parallel_{H_K^{0,-N}})$. Let $H_K^{0,-N}(P^r)$ denote the Hilbert space of elements in $D(P^r)$, r $\geq 0$, such that $\mid \cdot \mid_{r,N} < \infty$. The argument above applied to the spaces $H_K^{0,-N}(P^r)$ gives, for $\mathcal{U}_{-N}$ as in section 3.2

$$\sup_K \mid D^\alpha f *'' \mathcal{U}_{-N}(y) \mid \leq C \mid f \mid_{r,N} \leq C'_\delta \exp(-\kappa \mid \lambda \mid^b) \otimes \delta_{x''} \mid f \mid_{0,N} \tag{43}$$

The constant in the last expression, is dependent on the choice of mollifier. Note that in section 3.1, we proved that $\parallel \cdot \parallel_{H_K^{s,-N'}(L)}$ is norm equivalent to $\parallel \cdot \parallel_{H_K^{p_s,-N}}$ and we can show that this implies $\parallel \cdot \parallel_{H_K^{s,-N'}(\mathcal{A}_L)}$ is norm equivalent to $\parallel \cdot \parallel_{H_K^{s,-N}(\mathcal{A}_P)}$, so the inequalities we can prove for $\mathcal{A}_P$ also holds for $\mathcal{A}_L$, after adjusting the order of the Sobolev space in the "bad" variable. Since according to section 3.2, the iteration of the operator is done to satisfy a condition on the "good" variable, we prefer to work with realizations of the operator $P$.

For $f \in H$ and for the resolution corresponding to a realization of the operator L, we can write $E(\lambda)f = f_1 + f_2 + \ldots$, for $\lambda \neq -\infty$. For the spectral family corresponding to the iterated operator, $E_r(\lambda)$ and r odd, we have $E_r(\lambda^r) = E(\lambda)$. Through (43) and the partial norm equivalence proved in section 3.1, $\parallel f \parallel_{H_K^{0,-N'}(L^r)} \leq C \mid f \mid_{r,N}$, we know that

$$\sup_K \mid D^\alpha E(\lambda) f \mid \leq C(K,\alpha,\delta) \exp(-\kappa \mid \lambda \mid^b) \otimes \delta_{x''} \parallel f \parallel_H$$

for $f \in H$.

**The spectral kernel.** Assuming the operator $\mathcal{A}_{L^r}$, $\mathcal{U}_{0,-N}$-partially formally self-adjoint, we can construct the spectral kernel, in the Hilbert space $H_K^{0,-N}$. (Since we also assume that the operator $L^r(y,D_y) = \sum_{j=1}^t P_{j,(r)}(y,D_{y'})Q_{j,(r)}(D_{y''})$, is formally self-adjoint in $L^2$, this means a requirement that the operators $P_{j,(r)}(y,D_{y'})$ commute with the weight operators). The first estimate in (43), gives a bound for the "resolution ", $E_N$ on $H_K^{0,-N}$,

$$\sup_{\overline{F_4}} \mid D^\alpha E_N(\lambda)g \mid\leq C \mid g \mid_{0,N} \tag{44}$$

Define $T_{\lambda,N}^{(\alpha)}(x)g = D^\alpha E_N(\lambda)g(x)$, as in [16] (the derivatives are here taken in distribution sense). Schwartz kernel theorem (and for $T_{\lambda,N}^{(\alpha)}(x)$ interpreted as an evaluation functional, Riesz's representation theorem), gives existence of a $f_\lambda^{(\alpha)}(x,\cdot) \in H_K^{0,-N}$, such that $T_{\lambda,N}^{(\alpha)}(x)g = (g, f_\lambda^{(\alpha)}(x,\cdot))_{0,-N}$, for $x \in K$ and $g \in H_K^{0,-N}$ . We then have an implicitly defined spectral function in $L^2(\mathbf{R}^\nu)$, $e_\lambda(x,y) = f_\lambda *_y'' \mathcal{U}_{-N}\ (x,y)$, on a compact set. Note that $e_\lambda$ does not necessarily have compact support. The situation outside the compact set is dealt with in the end of this section. On the other hand, if $e_\lambda$ is the spectral function constructed on a compact set in $L^2$, our spectral kernel is given by $f_\lambda^{(\alpha)}(x,y) = (1-\Delta_{y''})^{N/2} e_\lambda^{(\alpha)}(x,y)$.

For the domain of definition to $T_{\lambda,N}$, we note that the kernel in $H_K^{0,-N}$, is defined as

$$f_\lambda(x,\cdot) = \begin{cases} f_\lambda^L & x \in K \\ f_\lambda^M & x \notin K \end{cases}$$

where $f_\lambda^L, f_\lambda^M$, are the kernels corresponding to the operators $L$ and $M$ respectively. Further, in the case where $x \in K$, we have

$$f_\lambda(x,y) = \begin{cases} f_\lambda^L(x,y) & y \in K \\ f_\lambda^M(x,y) & y \notin K \end{cases}$$

If in the scalar product $(f,g)_{0,-N}$, the weight is brought to one side, which would give an equivalent definition of the space $H^{0,-N}$, that is for $g \in L_K^2$, $(g,f)_{0,-N}^* = (g,(1-\Delta_{y''})^{-N}f)_{0,0} = (g,f)_{0,-N}$, then $T_{\lambda,N}$ can be defined on $L_K^2$. For an element $g \in L^2(\mathbf{R}^\nu)$, we have that $g = g_1 + g_2$, with $g_1 \in L_K^2$. So

$$(g, f_\lambda(x,\cdot))_{0,-N/2}^* = (g_1, f_\lambda^L(x,\cdot))_{0,-N/2}^* + (g_2, f_\lambda^M(x,\cdot))_{0,-N/2}^*$$

and we see that $T_{\lambda,N}$ is defined on $L^2(\mathbf{R}^\nu)$.

Now define a mapping from $\mathcal{D}$ into $\mathcal{D}'$ with kernel $e_\lambda^{(\alpha)}(x,y) \in \mathcal{D}'(\mathbf{R}^\nu \times \mathbf{R}^\nu)$, $T_\lambda^{(\alpha)}(x)f = D^\alpha E(\lambda)f(x)$ . For the partial regularity, we note that for any test function $\varphi$, if $\psi = (1-\Delta_{y''})^{-N}\varphi$, then using (38), we get

$$\sup_K \mid D^\alpha E_\lambda *'' \psi f \mid\leq C_{N,\alpha} \parallel \varphi \parallel_{L^1} \parallel f \parallel_H$$

the constant may depend on $\lambda_1, \lambda_2$, but it is not dependent on the choice of $\varphi$. For the partially regularized resolution, we have the following exponential estimate,

$$\mid T_\lambda^{(\alpha)}(x)f \mid\leq C(x,\alpha,\delta)\exp(-\kappa \mid \lambda \mid^b) \otimes \delta_{x''} \parallel f \parallel_H \tag{45}$$

for $x$ in $\mathbf{R}^\nu$, for $f \in H$ and $\lambda$ negative. The constant is again dependent on the way we mollify. Further $\parallel e_\lambda^{(\alpha)}(x,\cdot) \parallel\leq C(x,\alpha,\delta)\exp(-\kappa \mid \lambda \mid^b) \otimes \delta_{x''}$, for $x$ in $\mathbf{R}^\nu$. Just as in connection with the representation (41), we note that the estimate

implies $\| e_\lambda^{(\alpha)}(x',\varphi_\delta,y',\psi_\delta) \| \leq C(x,\alpha,\delta)\exp(-\kappa \mid \lambda \mid^b)C_{\varphi\otimes\psi}$. The estimate (44) still holds for the weighted spectral function $e_\lambda^{(\alpha)}(x,\cdot)$.

In the estimate following (45) above, if $\tilde{x}$ a point sufficiently close to $x$, we must have $\varphi(\tilde{x}'')$ arbitrarily close to $\varphi(\widetilde{x}'')$, which means, for $\lambda$ sufficiently large, that $\{e_\lambda^{(\alpha)}(\cdot,\varphi_\delta(\cdot),y',\psi_\delta(y''))\}$ is equicontinuous in $x$, for every $\alpha,\varphi,\psi$ and $y$. This holds for every $x \in \mathbf{R}^\nu$ and through a sequence $y_n \to y$, implies continuity in $(x,y)$ for the family of functions.

**Remark** (2): According to Proposition 19.0.1 .2 and (33), we have that

$$g_\lambda^{(\alpha',\beta')}(x,y) = O(1)\exp(-\kappa \mid \lambda \mid^b) \otimes (1+q)\delta_{x''} + O((R_\lambda^{\#})^{(\alpha',\beta')}(x,y))$$

Had we in (41) instead used a finite development of $g_\lambda$ such that, say $\deg q = k$, then for appropriate test functions, we would get the same estimates $B'_\lambda(x,y) = O(1)\exp(-\kappa \mid \lambda \mid^b) \otimes \delta_{x''}$. Since $g_\lambda(x,y',\cdot) \in \mathcal{D}'^F(\mathbf{R}^m)$, we can let $k$ go to infinity, which would allow an infinite development of $g_\lambda$ and we would still get the same estimate in (45), however we will not develop this approach any further.

Also

$$\mid D_y^{\beta'} e_\lambda^{(\alpha')}(x',y') \mid \leq C_{\beta'}\exp(-\kappa \mid \lambda \mid^b) \parallel e_\lambda^{(\alpha')}(x',\cdot) \parallel' = \tag{46}$$

$$= O(1)\exp(-\kappa_1 \mid \lambda \mid^b)$$

on compact sets in $\mathbf{R}^n$ and for $x'$ in $\mathbf{R}^n$. This is interpreted as regularity in $\mathbf{R}^n$, that is in the "good" variable. The notation $\| \cdot \|'$, indicates that the norm is taken over $\mathbf{R}^n$.

Outside $K$ $(\subset \Omega)$, the operator is $M(D')$ (we can assume $M(\xi') > 1$ for all $\xi'$). The spectral kernel then becomes, $f_\lambda(x,y) = \widetilde{f_\lambda}(x-y)$, where $\widetilde{f_\lambda}(z) = k_\lambda(z') \otimes \delta_0(z'')$ and

$$k_\lambda(z') = (2\pi)^{-n}\int_{M(\xi')<\lambda} e^{iz'\cdot\xi'}d\xi'$$

and we know $k_\lambda \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$. The following theorem and the preceding argument is close to [17] Theorem 1.

**Theorem 26.1.1.** *Assuming the operator partially formally self adjoint, we can for every real $\lambda$, implicitly define an element $e_\lambda(x,\cdot) \in H$, which mollified with test functions with support sufficiently close to the origin, is in $C^\infty(\mathbf{R}^\nu \times \mathbf{R}^\nu)$. Further $e_\lambda^{(\alpha,0)}(x,\cdot)$ (distribution sense derivatives) is in $H$, for all $\alpha$,$x \in \mathbf{R}^\nu$ and*

$$E(\lambda)u(x) = \int e_\lambda(x,y)u(y)dy \quad \text{for } u \in H \text{ and } x \in \mathbf{R}^\nu$$

*For appropriate test functions, we have the estimates $D_y^\beta e_\lambda^{(\alpha,0)}(x,y) = O(1)\exp(-\kappa \mid \lambda \mid^b) \otimes \delta_{x''}$, uniformly on compact sets in $\mathbf{R}^n \times \mathbf{R}^n$ and $\| e_\lambda^{(\alpha,0)}(x',\varphi_\delta,\cdot,\psi_\delta(\cdot)) \| = O(1)\exp(-\kappa \mid \lambda \mid^b)$, uniformly on compact sets in $\mathbf{R}^\nu$, as $\lambda \to -\infty$, for $x$ in $\mathbf{R}^\nu$.*

Note that the theorem implies the representation:

$$E(\lambda)v(x) = \int f_\lambda *''_y \mathcal{U}_{-N}(x,y)v(y)dy \quad \text{for} \quad v \in L^2, \quad x \in \mathbf{R}^\nu$$

**Asymptotic behavior of the spectral kernel.** In this section we study the regularizations $\tilde{e}_\lambda(x,y) = e_\lambda *_x'' \varphi *_y'' \psi(x,y)$, where $\varphi, \psi$ are appropriate test functions, of the spectral kernel corresponding to the partially formally hypoelliptic operator with variable coefficients $L$, first with the assumption that $L \geq I$. The corresponding integral operator is denoted $\tilde{E}_\lambda$. We first note some immediate results (derived from [1]):

For an interval $\Delta$ in $\mathbf{R}$ and $f \in L^2(\mathbf{R}^\nu)$, we have

$$\mathrm{var}_\Delta \delta^{\alpha\beta}\tilde{e}_\lambda(x,y) \leq \Big[\mathrm{var}_\Delta \delta^{\alpha\alpha}\tilde{e}_\lambda(x,x)\mathrm{var}_\Delta \delta^{\beta\beta}\tilde{e}_\lambda(y,y)\Big]^{1/2}$$

Here $\mathrm{var}_\Delta$ denotes the variation over the interval $\Delta$ and $\delta^{\alpha\beta}$ derivation in x and y. Further $\mathrm{var}_\Delta \delta^{\alpha}\tilde{e}_\lambda(f,y) \leq\mid \tilde{E}_\Delta f \mid \mathrm{var}_\Delta \delta^{\alpha\alpha}\tilde{e}_\lambda(y,y)^{1/2}$ for $f \in L^2(\mathbf{R}^\nu)$.

For test functions $\psi = \overline{\varphi}$, we have $\delta^{\alpha\alpha}\tilde{e}_\lambda(x,x) \geq 0$ and it is a non-decreasing function of $\lambda$. The function $\delta^{\alpha\beta}\tilde{e}_\lambda(x,y)$ is locally of bounded variation as a function of $\lambda$, for $(x,y) \in \mathbf{R}^\nu \times \mathbf{R}^\nu$

For a realization of the operator, that is a self-adjoint spectral operator $\mathcal{A}_L$, we have that the resolvent operator $\mathrm{G}(\lambda) = (\mathcal{A}_L - \lambda I)^{-1}$ is a bounded spectral operator on $L^2(\mathbf{R}^\nu)$ and

$$G(\lambda) = \int_1^\infty \frac{dE_\mu}{\mu - \lambda} \qquad \lambda < 1$$

$\tilde{G}(\lambda)$ corresponding to the regularized spectral kernel, can also be represented as an integral operator with kernel $\tilde{G}_\lambda(x,y)$

$$\tilde{G}(\lambda)f(x) = \int \tilde{G}_\lambda(x,y)f(y)dy = \int\int(\mu-\lambda)^{-1}d\tilde{e}_\mu(x,y)f(y)dy$$

where according to Stieltjes formula, $\tilde{E}_\mu f(x) = \int \tilde{e}_\mu(x,y)f(y)dy$ and $f \in L^2(\mathbf{R}^\nu)$

Assuming we can prove an a priori estimate

$$\sup_{x,y\in K} \mid \tilde{e}_\lambda^{(\alpha,\beta)}(x,y) \mid\leq C_{K,\alpha,\beta}(1+\lambda)^c \qquad \lambda \geq 1$$

then with condition the constant $c$ is $< 1$, we have that $\tilde{G}_\lambda(x,y)$ is continuous on $\mathbf{R}^\nu \times \mathbf{R}^\nu$.

Proof of the estimate: Assume with notation as in section 26.1, that $\varphi \in H(0,\lambda)$. According to the inequality (39) in that section, we have for $kr > n/2 + m$ $(\mid \beta \mid\leq m)$

$$\sup_{y\in K} \mid D_y^\beta \widetilde{\varphi}(x,y) \mid\leq C_{K,\beta}(\parallel P^r\varphi \parallel_H + \parallel \varphi \parallel_H) \qquad x \in \mathbf{R}^\nu \tag{47}$$

The inequality for a $\lambda$-approximative eigenvector to $P^r$ and 0, where $r$ is assumed odd, gives

$$\parallel P^r\varphi \parallel_H\leq \lambda \parallel \varphi \parallel_H$$

Using the relation $E(\lambda) = E_r(\lambda^r)$ and (47), we get for the resolution corresponding to the non-iterated operator,

$$\sup_{y\in K} \mid D_y^\beta \widetilde{\varphi}(x,y) \mid\leq C_{K,\beta}(1+\lambda^{1/r}) \parallel \varphi \parallel_H\leq C'_{K,\beta}(1+\lambda)^{1/r} \parallel \varphi \parallel_H \quad x \in \mathbf{R}^\nu$$

Finally, again using chapter 26.1

$$\sup_{x,y\in K} \mid D_x^\alpha \tilde{E}(\lambda) D_y^\beta \widetilde{\varphi} \mid \leq C_{K,\alpha,\beta}(1+\lambda)^{1/r} \parallel \varphi \parallel_H$$

and by choosing $\widetilde{\varphi}$ as the regularized spectral function $\tilde{e}_\lambda(x,y)$, knowing that this function is in $L^2$, the estimate follows.□

**Estimate for the Green kernel.** Using partial integration, we can write for $\mid \alpha' + \beta' \mid \leq M$ , for all $\alpha'', \beta''$ and for $f, g \in C_0^\infty$

$$(48) \qquad (\tilde{G}_\lambda^{(\alpha,\beta)} f, g) = \int_1^\infty (\mu-\lambda)^{-1} d(\tilde{E}_\mu^{(\alpha,\beta)} f, g) \qquad \lambda < 1$$

For the Green kernel corresponding to the operator with constant coefficients $P^x(D_{y''}) - \lambda$

$$(49) \quad \tilde{G}_{x,\lambda}^{(\alpha,\alpha)}(x,x) = \int_1^\infty (\mu-\lambda)^{-1} d\tilde{e}_{x,\mu}^{(\alpha,\alpha)}(x,x) = \Big[(iD)^{\alpha''}\varphi\Big]\Big[(iD)^{\alpha''}\psi\Big] \int \frac{\xi'^{2\alpha'} d\xi'}{\mathrm{Re}P^x(\xi') - \lambda}$$

where the integral is finite, for all $\alpha''$ and $s > n + 2 \mid \alpha' \mid$ and where we have used that $e_{x,\lambda}$ must be a tensor product with $\delta_{x''}$.

A modification of Nilssons's article (cf.[18]), gives that we can use an estimate of the regularized fundamental solution $\widetilde{g}_\lambda$, to produce an estimate of the difference between the resolvent operators corresponding to the operator $\mathrm{Re}P^x - \lambda$ and the variable coefficients operator respectively, in terms of the first operator. More precisely

$$(50) \qquad \tilde{G}_{x,\lambda}^{(\alpha,\alpha)}(x,x) - \tilde{G}_\lambda^{(\alpha,\alpha)}(x,x) = O(1) \mid \lambda \mid^{-c} \tilde{G}_{x,\lambda}^{(\alpha,\alpha)}(x,x) \quad \lambda \to -\infty$$

The left side in (50) can be written $\int_1^\infty (t-\lambda)^{-1} d\sigma(t)$, where $\sigma(t) = \tilde{e}_{x,t}^{(\alpha,\alpha)}(x,x) - \tilde{e}_t^{(\alpha,\alpha)}(x,x)$, which is a monotone non-decreasing function of $t$.

**Estimate for the spectral kernel.** Tauberian theory applied to $\sigma(\mu)$, leads to an estimate of the regularized spectral function corresponding to the partially formally hypoelliptic operator, in terms of the regularized spectral function corresponding to the operator $\mathrm{Re}P^x - \lambda$. Note that

$$\tilde{e}_{x,\lambda}^{(\alpha,\alpha)}(x,x) = \Big[(iD)^{\alpha''}\varphi\Big]\Big[(iD)^{\alpha''}\psi\Big] \int_{\mathrm{Re}P^x(\xi')<\lambda} \xi'^{2\alpha'} d\xi'$$

where $\lambda \in \mathbf{R}$. Assuming $\psi = \overline{\varphi}$, we can prove (cf.[18]),

**Lemma 26.1.** *There is a complex constant C, a rational number a and an integer t,* $0 \leq t \leq n-1$, *such that*

$$\tilde{e}_{x,\lambda}^{(\alpha,\alpha)}(x,x) = C(1+o(1))\lambda^a (log\lambda)^t \quad \lambda \to \infty$$

*Further* $\tilde{e}_{x,\lambda}^{(\alpha,\alpha)}(x,x)$ *is infinitely differentiable for large* $\lambda$ *and*

$$\frac{d\tilde{e}_{x,\lambda}^{(\alpha,\alpha)}(x,x)}{d\lambda} = o(1)\lambda^{a-1}(log\lambda)^t$$

*as* $\lambda \to \infty$

**Lemma 26.2.** *For a positive constant c as in (50), we have*

$$\sup_{\lambda \leq \mu \leq \lambda + \lambda/(clog\lambda)} \int_\lambda^\mu d\sigma(w) = o(1)\lambda^a (log\lambda)^{r-1}$$

Proof: (cf.[26]) Immediately from Lemma 26.1

$$\frac{d}{dt}\tilde{e}^{(\alpha,\alpha)}_{x,t}(x,x) = o(1)t^{a-1}(\log t)^r \quad t \to \infty$$

The properties of $\sigma(t)$ imply $\int_t^\mu d\tilde{e}_t^{(\alpha,\alpha)}(x,x) \geq 0$. If $t \leq \mu \leq t + t/c\log t$, we get $\int_t^\mu d\sigma(t) \leq \int_t^\mu d\tilde{e}^{(\alpha,\alpha)}_{x,t}(x,x) \leq C\int_t^\mu t^{a-1}(\log t)^r dt \leq C'(t/\log t)t^{a-1}(\log t)^r = C't^a(\log t)^{r-1}\square$

Immediately from (49), (50) and Lemma 26.1, we get

$$\int_1^\infty \frac{d\sigma(\mu)}{\mu+\lambda} = O(1) \mid \lambda \mid^{-c} \lambda^{a-1}(\log\lambda)^r \quad \lambda \to \infty$$

That is if we use the Stieltjes transform, we see that

$$\int_1^\infty \frac{d\widetilde{e}^{(\alpha,\alpha)}_{x,\mu}(x,x)}{\mu+\lambda} = \mathcal{L}(\mathcal{L}(\frac{d}{d\mu}\widetilde{e}^{(\alpha,\alpha)}_{x,\mu}(x,x)))$$

and a well known result in Abelian theory of the Laplace transform, gives that $\overline{\lim}_{\mu\to\infty}\mathcal{L}(\frac{d}{d\mu}\widetilde{e}^{(\alpha,\alpha)}_{x,\mu}) \leq \overline{\lim}_{\mu\to\infty}(\frac{d}{d\mu}\widetilde{e}^{(\alpha,\alpha)}_{x,\mu})$ and the result follows since $\mu \leq \lambda$. This means that the conditions in Ganelius Tauberian theorem (cf.[5]) are satisfied for the operator $L \geq I$. However this restriction can be discarded if we study instead the operator $\mathcal{A}_L^r + kI$, for $k, r$ sufficiently large, both numbers are assumed $\geq 0$, $r$ is an even integer and $k$ is real, since this translation does not effect the asymptotic behavior. For details we refer to [17].

**Theorem 26.3.** *For every multi-index $\alpha$ and every $x \in W$, $W$ a compact set where the operator $L_\lambda$ is assumed partially formally hypoelliptic, we have*

$$\tilde{e}^{(\alpha,\alpha)}_\lambda(x,x) = (1 + o(1)(log\lambda)^{-1})\tilde{e}^{(\alpha,\alpha)}_{x,\lambda}(x,x) \qquad \lambda \to \infty$$

[27]

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